

A P-Stable Linear Multistep Method for Direct Solution of General Third Order Ordinary Differential Equations

D.O.Awoyemi
Federal university of Technology Akure, Nigeria

Abstract

A P-stable linear multistep method for the direct solution of third order initial value problems of ordinary differential equations is considered. The approach for the development of this method is essentially based on collocation of the differential systems generated from the basis function. A predictor for the evaluation of y_{n+k} for an odd $k \geq 3$ in the main method is also proposed. The two resulting methods, the corrector and the predictor, are P-stable for $k = 3$. These methods as a block are tested on a number of problems to show their efficiency. When the methods (the corrector and the predictor) are evaluated at $x = x_{n+3}$, identical schemes are obtained as special cases of the methods, while the set of the derivatives obtained from the corrector are different from those obtained from the predictor.

Key Words: Linear multistep method; P-stable; third order IVPS of ODEs; Stiff and non-stiff initial value problems; corrector; predictor

1.0 Introduction

In this article, we consider general third order initial value problems of the form

$$y''' = f(x, y, y', y''), y(a) = y_0, y'(a) = n_1, y''(a) = n_2, a, y, f \in R$$

For an odd stepnumber $k \geq 3$

This class of problems is important for their applications in Science and Engineering especially in Biological Sciences and Control Theory.

Efforts have been made by eminent scholars to solve higher order initial value problems especially the special second order ordinary differential equations by different methods. Lambert (1973), Enright (1974), Jeltsch (1976), Brown (1974, 1977), Wanner et al (1978), Jeltsch et al (1978), Twizell and Khaliq (1981, 1984) proposed independently a method called Mutiderivative Method to solve second order initial value problems. They concluded that Mutiderivative methods give high accuracy and possess good stability properties when used to solve first order initial value problems. Twizell and Khaliq (1984), however, proposed a class of P-stable two-step higher derivative formulas for special second order initial value problems by adopting pade approximation technique. Onumanyi and Ortiz (1982) proposed the Tau method to solve higher order initial and boundary value problems, Wright et al (1991) adopted a method called the mesh selection

in collocation to solve second and fourth order boundary value problems. Awoyemi (1996, 1999, 2001) used Mutiderivative Collocation approach to solve directly general second order initial value problems of ordinary differential equations. There are still other methods proposed for higher order initial and boundary value problems that could not be accommodated in this paper for lack of space. In most of the methods mentioned above, reduction of the problems into a system of first order ODEs was adopted and methods designed for first order ODEs were used to solve them (see for instance Lambert (1973), Brown 1974), Jeltsch (1976), Enright (1974) and Twizell and Khaliq (1981, 1984)).

We have shown in Awoyemi (1999,2001) that reduction of higher order initial value problems to a system of first order equations has serious problems, the most notable being non economization of computer time, computational burden and costs implications. For example we need to determine the eigenvalue λ from the Jacobian of a system of first order equations in order to find the value of $\bar{h} = h\lambda$ which is expected to lie inside the region of absolute stability of the method to ensure consistency and good accuracy of the method. All these are avoided in our method which solves eqn. (1.1) directly without reducing it to a system of first order equations.

2. The Method

It is a common practice to construct the basis function for the proposed method from the given problem we are solving (see for instance Awoyemi (1993, 1996, 1999, 2001)).

Thus we define an operator L from equation (1.1) as:

$$L = I + \sum_{i=1}^k \frac{d^i}{dx^i} \quad (2.1)$$

where k is the stepnumber of the method. For the construction of this method, k is taken to be odd with the values $k \geq 3$.

From this definition, we write equation (1.1) in the form

$$Ly = \left(I + \sum_{i=1}^k \frac{d^i}{dx^i} \right) y = f(x, y, y^i, y^n) + \left(I + \sum_{i=1}^{k-1} \frac{d^i}{dx^i} \right) Y \quad (2.2)$$

Secondly, we define a basis function as

$$L\Psi(x) = x^j, j \geq 0 \quad (2.3)$$

which leads to a recurrence relation

$$\Psi_j(x) = x^j - (j\Psi_{j-1}(x) + j(j-1)\Psi_{j-2}(x) + \dots + j(j-1)(j-2)\dots(j-m+1)\Psi_{j-m}(x)), j = 0(1)m \quad (2.4)$$

Where m is the degree of $\Psi_j(x)$ with the value $m = 2k$.

Thus in this article, an approximate solution of the form

$$y(x) = \sum_{j=0}^m a_j \Psi_j(x) \quad (2.5)$$

is proposed whose higher derivatives are listed as follows

$$\begin{aligned} y^i(x) &= \sum_{j=0}^m j a_j \Psi_{j=1}(x), \\ y^i(x) &= \sum_{j=0}^m j(j-1) a_j \Psi_{j=2}(x), \\ y^m(x) &= \sum_{j=0}^m j(j-1)(j-2) a_j \Psi_{j=3}(x), \end{aligned} \quad (2.6)$$

Where $\Psi_j(x)$ is the required basis function and $a_j, j = 0, 1, 2, \dots, m$ are the parameters to be determined.

On substitution equations (2.5) and (2.6) into equation (1.1), we have

$$\sum_{j=0}^m j(j-1)(j-2) a_j \Psi_{j=3}(x) = f(x, y(x), y^i_k(x) y^n_k(x)) \quad (2.7)$$

Where $y(x), y^i(x)$ and $y^n(x)$ have the values indicated in equations (2.5) and (2.6). Collocating equation (2.7) at the grid points, $x = x_{n+j}, I = 0(1)k$ and interpolating equation (2.5) at $x = x_{0+i}, i = 0(1)k - 1$ yield the following system of equations

$$\sum_{j=0}^m j(j-1)(j-2) a_j \Psi_{j=3}(x_{n+j}, i = 0(1)k) \quad (2.8)$$

$$\sum_{j=0}^m a_j \Psi_j(x_{n+i}) = y_{n+i}, i = 0(1)k - 1 \quad (2.9)$$

Solving equations (2.8) and (2.9) by matrix or Gaussian elimination method to obtain the values of the parameters a_j 's which when substituted into (2.5) give a scheme expressed in the form

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \sum_{j=0}^k \beta_j(x) f_{n+j} \quad (2.10)$$

Where $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j}, y''_{n+j})$.

The next step is to specify our k which for this method is odd with the values $k \geq 3$ to enable us determine the value of m . Thus if for example we take $k \geq 3$, then $m = 2k = 6$, then equation (2.10) becomes

$$y(x) = \sum_{j=0}^{k-2} \alpha_j(x) y_{n+j} + \sum_{j=0}^3 \beta_j(x) f_{n+j} \quad (2.11)$$

The coefficients $\alpha_j(x)$ and $\beta_j(x)$ in equation (2.11) and their first and second derivatives are thus given by letting

$$t = (x - x_{n+2}) / h \quad (2.12)$$

as follows:

$$\begin{aligned} \alpha_0(t) &= \frac{1}{2}(t^2 + t), \\ \alpha_1(t) &= -(t^2 + 2t), \\ \alpha_2(t) &= \frac{1}{2}(t^2 + 3t + 2) \\ \beta_0(t) &= \frac{h^3}{720}(-t^6 + 5t^4 - 4t^2), \\ \beta_1(t) &= \frac{h^3}{720}(3t^6 + 6t^5 - 30t^4 + 207t^2 + 174t), \\ \beta_2(t) &= \frac{h^3}{720}(-3t^6 - 12t^5 + 15t^4 + 120t^3 + 168t^2 + 72t), \\ \beta_3(t) &= \frac{h^3}{720}(t^6 + 6t^5 + 10t^4 - 11t^2 - 6t), \end{aligned} \quad (2.13)$$

First derivatives of (2.13)

$$\begin{aligned} \alpha'_0(t) &= \frac{1}{2h}(2t + 1), \\ \alpha'_1(t) &= \frac{-2}{h}(t + 1), \\ \alpha'_2(t) &= \frac{1}{2h}(2t + 3), \\ \beta'_0(t) &= \frac{h^3}{720}(-6t^5 + 20t^3 - 8t), \\ \beta'_1(t) &= \frac{h^3}{720}(18t^5 + 30t^4 - 120t^3 + 414t + 174), \\ \beta'_2(t) &= \frac{h^3}{720}(-18t^5 - 60t^4 + 60t^3 + 360t^2 + 336t + 72t), \\ \beta'_3(t) &= \frac{h^3}{720}(6t^5 + 30t^4 + 40t^3 - 22t - 6), \end{aligned} \quad (2.14)$$

Second Derivative of (2.13)

$$\begin{aligned}
\alpha^n_0(t) &= \frac{1}{h^2} \\
\alpha^n_1(t) &= \frac{2}{h} \\
\alpha^n_2(t) &= \frac{1}{h^2} \\
\beta^n_0(t) &= \frac{h}{720}(-30t^4 + 60t^2 - 8), \\
\beta^n_1(t) &= \frac{h}{720}(90t^4 + 120t^3 - 360t^2 + 414), \\
\beta^n_2(t) &= \frac{h}{720}(-90t^4 - 240t^3 + 180t^2 + 720t + 336), \\
\beta^n_3(t) &= \frac{h}{720}(30t^4 + 120t^3 + 120t^2 - 22),
\end{aligned} \tag{2.15}$$

When $t = 1 \rightarrow x = x_{n+3}$, we have from equations (2.13), (2.14) and (2.15)

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^2}{2}(f_{n+2} + f_{n+1}), \tag{2.16}$$

With order $p=4$, Error Constant $C_2 = \frac{1}{240}$

and interval of periodicity $x(\theta) = (0, \infty)$ (see Fatunla (1988))

$$y^i_{n+3} = \frac{1}{2h}(5y_{n+2} - 8y_{n+1} + 3y_n) + \frac{h^2}{720}(48f_{n+3} + 750f_{n+2} + 516f_{n+1} + 6f_n) \tag{2.17}$$

$$y^n_{n+3} = \frac{1}{h^2}(5y_{n+2} - 2y_{n+1} + y_n) + \frac{h}{720}(248f_{n+3} + 906f_{n+2} + 264f_{n+1} + 22f_n) \tag{2.18}$$

$$f_{n+j} = f(x_{n+j}, y_{n+j}, y_{n+j}, y^n_{n+j}), j = 0, 1, 2, 3. \tag{2.19}$$

3. The Predictor

In developing the predictor, we employ the same collocation procedure adopted in the main method, which yields the method

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \sum_{j=0}^{k-1} \beta_j(x) f_{n+j} \tag{3.1}$$

Again put $k=3$ and $t = (x - x_{n+2})/h$, the coefficient $\alpha_j(x)$ and $\beta_j(x)$ and their first and second derivatives are listed as follows:

$$\begin{aligned}
\alpha_0(t) &= \frac{1}{2}(t^2 + t), \\
\alpha_1(t) &= -(t^2 + 2t), \\
\alpha_2(t) &= \frac{1}{2}(t^2 + 3t + 2t), \\
\beta_1(t) &= \frac{h^3}{24}(-t^4 + 7t^2 + 6t), \\
\beta_2(t) &= \frac{h^3}{24}(t^4 + 4t^3 + 5t^2 + 2t),
\end{aligned} \tag{3.2}$$

First Derivative of (3.2);

$$\begin{aligned}
\alpha^1_0(t) &= \frac{1}{2h}(2t + t), \\
\alpha^1_1(t) &= \frac{-2}{h}(t + 1), \\
\alpha^1_2(t) &= \frac{1}{2h}(2t + 3), \\
\beta^1_1(t) &= \frac{h^3}{24}(-4t^3 + 14t + 6), \\
\beta^1_2(t) &= \frac{h^3}{24}(4t^3 + 12t^2 + 10t + 2),
\end{aligned} \tag{3.3}$$

Second Derivative of (3.2)

$$\begin{aligned}
\alpha^n_0(t) &= \frac{1}{h^2}, \\
\alpha^n_1(t) &= \frac{-2}{h^2}, \\
\alpha^n_2(t) &= \frac{1}{h^2}, \\
\beta^n_1(t) &= \frac{h}{24}(-12t^2 + 14), \\
\beta^n_2(t) &= \frac{h}{24}(12t^2 + 24t + 10),
\end{aligned} \tag{3.4}$$

We put $t = 1 \Rightarrow x = x_{n+3}$ in (3.2), (3.3) and (3.4) to obtain

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{2}(f_{n+2} + f_{n+1}) \tag{3.5}$$

with order $P = 4, C_7 = \frac{1}{240}$ interval of periodicity $x(\theta) = (0, \infty)$.

Thus the method is P stable.

Furthermore, the first and second derivatives of (3.5) are computed as follows:

$$y'_{n+3} = \frac{1}{2h}(5y_{n+2} - 8y_{n+1} + 3y_n) + \frac{h^2}{6}(7f_{n+2} + 4f_{n+1}) \quad (3.6)$$

$$y''_{n+3} = \frac{1}{h^2}(y_{n+2} - 8y_{n+1} + y_n) + \frac{h}{12}(23f_{n+2} + f_{n+1}) \quad (3.7)$$

Finally, we use Taylor series expansion to calculate the values of y_{n+1} and y_{n+2} and their first and second derivatives at $x = x_n$ in (2.16) as follows:

$$\begin{aligned} y_{n+1} &= y(x_{n+1} + h) = y(x_n) + thy'(x_n) + \frac{(th)^2}{2!}y''(x_n) + \frac{(th)^3}{3!}f_n + \frac{(th)^4}{4!}f'_n + \frac{(th)^5}{5!}f''_n + \dots \\ y'(x_{n+1} + h) &= y'(x_n) + th y''(x_n) + \frac{(th)^2}{2!}f_n + \frac{(th)^3}{3!}f'_n + \frac{(th)^4}{4!}f''_n + \dots \\ y''(x_{n+1} + h) &= y''(x_n) + th f_n + \frac{(th)^2}{2!}f'_n + \frac{(th)^3}{3!}f''_n + \dots \end{aligned} \quad (3.8)$$

where $f_n = f(x_n, y_n, y'^1_n, y''_n)$, $f^i = f^i(x_n, y_n, y'^1_n, y''_n)$, $i = 1, 2$

Furthermore, we put (2.18) in the form

$$f = f(x, y, y', y'') \quad (3.9)$$

and find f' and f'' by partial derivative technique as follows:

$$f' = \frac{df}{dx} = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} + f \frac{\partial f}{\partial y''}, y^i = \frac{d^i y}{dx^i}, i = 1, 2, \dots \quad (3.10)$$

$$f'' = \frac{d^2 f}{dx^2} = 2(Ay^i + By^n + Cf) + D + E \quad (3.11)$$

where

$$A = \frac{\partial^2 f}{\partial x \partial y} + y'' \frac{\partial^2 f}{\partial y \partial y^i} + f \frac{\partial^2 f}{\partial y \partial y^n}$$

$$B = \frac{\partial^2 f}{\partial x \partial y^i} + f \frac{\partial^2 f}{\partial y \partial y^n}$$

$$C = \frac{\partial^2 f}{\partial x \partial y^n}$$

$$D = y'' \frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y^i} + f' \frac{\partial f}{\partial y^n}$$

$$E = \frac{\partial^2 f}{\partial x^2} + (y^i)^2 \frac{\partial^2 f}{\partial y^2} + (y^n)^2 + (f)^2 \frac{\partial^2 f}{\partial y^{n^2}}.$$

4. Test Problems

(1) $y''' + 4y' = x, y(0) = y'(0) = 0, y''(0) = 1$

$$y(x) = \frac{3}{16}(1 - \cos x) + \frac{1}{8}x^2$$

(2) $y''' + y' = 0, y(0) = 0, y'(0) = 1, y''(0) = 2$

$$y(x) = 2(1 - \cos x) + \sin x$$

Numerical Solution of Test Problems

The following tables display maximum error values for $200 \leq nT \leq 800$ iterations.

Table 1
Problem 1

nT	X	y -Exact	y -Computed	Error
200	5.0000	0.3469825912D+01	0.3469821975D+01	3.94D-06
400	10.0000	0.1261098461D+02	0.1261098841D+02	3.80D-06
600	15.0000	0.2828357785D+02	0.2828358015D+02	2.29D-06
800	20.0000	0.5031255089D+02	0.5031253791D+02	1.30D-06

Table 2
Problem 2

nT	X	y -Exact	y -Computed	Error
200	5.0000	0.4737513544D+00	0.4737548867D+00	3.53D-06
400	10.0000	0.3134121947D+01	0.3134119698D+01	2.25D-06
600	15.0000	0.4169663666D+01	0.4169653815D+01	9.85D-06
800	20.0000	0.2096781127D+01	0.2096780496D+01	6.31D-07

5. Conclusion

A p -stable method for direct solution of general third order initial value problems is developed. The efficiency of this method is encouraging judging from the small error values recorded in the tables. Furthermore, the method is economical to implement on general problems, since there are only two functions to evaluate per iteration in both the corrector and the predictor methods.

Two test problems are considered to demonstrate the efficiency and p -stability of the new method. Even though the computer numerical solutions are printed from $x = 0$ to $x = 20$ for $h = 0.025$, only the solutions at the integral values of $x = 5, 10, 15$ and 20 are shown in Tables 1 and 2 with the exact solution (y -Exact), the

computed solution (**y-Computed**) to 10 decimal places, while the errors (**Error**) are shown to 2 decimal places.

The efficiency of the method is seen in the Tables by the small error values recorded at $x = 5, 10, 15$ and 20 , while the P -stability of the method is also seen by the values of x recorded, the maximum of which can be increased if desired. This is evident from the last error value recorded at $x = 20$.

Finally, it is interesting to note that while both corrector and predictor yield identical schemes of order five at $x = x_{n+3}$ (see equations (2.16) and (3.5)), the first and second derivatives of (2.16) are different from the first and second derivatives of (3.5) (see equations (2.17) and (2.18), (3.6) and (3.7) respectively).

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