H^{-1} Galerkin-Collocation and Quasi-Iterative Method for Boundary value problems

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Abstract

H^{-1} –Galerkin Method for the solution of differential equation was initiated by Richford Jr. and Wheeler [1] and was improved upon by Dupont [2]. Here, we first define an H^{-1} Galerkin method on a boundary value problem and then an iterative process that leads to super convergence results.

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1.0 Introduction

Define

$$(U,V) = \int \alpha(uv)dx, \; x \in \Omega$$

Then the H^{-1} Galerkin inner product is defined by

$$D^{-1}(U,V) = \int \Omega \left\lfloor \int (u)dx \int (v)dx \int dx \right\rfloor \tag{1}$$

Where D^{-1} is the inverse of the differential operator D and hence; an integral operator

Now consider the equation
$Lu = f$, in $1 = (a,b)$
And

$$U(a) = u(b) = 0 \tag{2}$$

Let, $z \in S^h$, be the H^{-1} -Galakin approximation for the exact solution u of equation (2). Then z satisfies the H^{-1} Galerkin equation

$$D^{-1}(L\ V) = D^{-1}(f,V), \forall \in S^h \tag{3}$$

Where S^h is defined by

$$S^h M_{-1}(\tau\, \Delta) = \{\forall / \forall \in P_\tau(I_j), j = 1,2,\ldots,N\} \tag{4}$$

$\Delta : a = X_0 < X_1 <,\ldots, X_N = b, 1_j = (X_{j-1}, X_j)$

If $L = L = \frac{d^2}{dx^2}$, then on integrating equation (3) twice by parts, we have

$$(LZ,V) = (Z,L'V) = (f,V) \tag{5}$$

for any $V \in H_0(1)$
$(L'$ is the adjoint of $L)$.
Since equation (3) holds for all

$$V \in H^2(1)^0 \in M_{k+2}{}'(r+2,\Delta) \qquad (6)$$

(See Richford and Wheeler [1]).
Note: The boundary conditions now shift to V; so that the smoothness condition on the trial space or solution space is relaxed. Since equation (5) is very much similar to the standard Galerkin equation (except that the boundary conditions now shift to V and that the operator 1^*, now operates on V), we expect the procedure for the error estimate to be similar, and hence the error estimate (see ref [2]).
Note' also that we now require $V \in M_{k+1}{}^0(r+2,\Delta)$ instead of S^h in (4). This is understandable since the function in S^h as defined by (4) can be discontinuous at the knots X_j and have values on I_j that are independent of values on $I_j, i = j,$ the attempted use of S^h as the test space provides no indication of the interaction of data on one subinterval with the solution on another.
Now, there are N subintervals and with polynomials of order $(r+3)$. There are $2(N-1)$ constraints in the interior of 1 and 2 boundary conditions.
Hence, the dimension D of $M^0{}_{k+1}(r+2,\Delta)$ is
$D = N(r+3) - 2(N-1) - 2$
$= Nr + 3N - 2N = Nr + nN = N(r+1$
which is the same as the dimension of S^y and is therefore finite – dimension. This is sufficient for existence of solution. The uniqueness of solution is assured by the following theorem by Richford and Wheeler [1].

2.0 Uniqueness and Existence of Solutions

Theorem 1. For h sufficiently small $\exists$ a unique solution Y_h

$$(Y_h, L'V) = (f, V), v \in N_h$$

Moreover,

$$|||Y_h - y||| < C|||y - y|||| < C|||y|||h^k{}_k, 0 < q < r+1$$

where $\underline{y}$ is the project of y defined by

$$(y - y, z) = 0 \ , \ z \in \mathop{X}_{j=1}^{m} P_r(1_j)$$

where, z is the H^{-1} Galerkin approximation for y.
Proof

$$(Ly, V) = (f, V), V \in N_h \qquad (7)$$

where

$$Ly = y'' + ay' + by = f, y(0) = y(1) = 0$$

And
$L'y = y'' + -(ay)' + by$

$$\begin{aligned}
(Ly,V) &= (y'',V) + (y',av) + (y,bv) \\
&= -(y',V') + y'V'_0 - (y,(av)') \\
&\quad + ayv\,|_0' + (y,bV) \\
&= (y,v'') - yv'|_0' + (y,-(av)') + (y,bV) \\
&= y,V''(aV)' + bv. \\
&= (y,L^* v)
\end{aligned}$$

i.e.

$(y,L^* y) = (f,v), v \in H^2(1) \cap H\,|_0^1\,(1)$

The H^{-1} Galerkin method is

$$(\overline{y}_a, L^* v) = (f,v), \quad v \in N_h \tag{9}$$

Then (3) – (2 gives

$$(Y_h - y, L^* v) \tag{10}$$

Now, let

$$L^* \phi = \in, x \in 1 \tag{11}$$

$\phi(0) = \phi(1) = 0.$

Then

$$(Y_h - y, L^* \phi) = (Y_h - y, \in)$$

then using (4), we have that $(Y_h - y,\ \in) = (Y_h - y, L^*(\phi - v))$

$$= (\overline{y} - y, (\phi - v)'') + (\overline{y} - y, a(\phi - v)')b(\overline{y}_h - y, \phi - v)$$

$$\| Y_h - y, \in \| < \left[c \,\| Y_h - y \| + c_1 h \,\| Y_h - y \| \right] |_2$$

$$\begin{aligned}
\| Y_h - y, \in \| \, \| \in \| &< c \,\| Y_h - y \| + c_1 h \,\| Y_h - y \| \in \| \\
&< (\| \overline{y} - y \| + c_1 h \,\| \overline{y} - y \|) \,\| \in \|
\end{aligned}$$

i.e.

$$\| Y_h - y \| < C_2 h \,\| \overline{y} - y \| < C_2 h \,\| h \,|_k^k \qquad 0 < q < r + 1$$

3.0 **Iterative Process**

We consider the m^{th} order differential equation.

$$\begin{aligned}
D^m u(x) &= F(x, u(1), Du(x), \ldots, D^{m-1} u(x)) \\
&= (Fu)(x),
\end{aligned} \tag{12}$$

with the boundary conditions:

$B_j u = c_j, \ \ i = 1,2,\ldots,m$

Where F is a real-valued function and $B_i, 1 = 1,2,\ldots,m.$

Some known sequence of conditions linear functional on $C^{m-1}(a,b)$. *And the* C_j are constants.

We assume that the problem (12) has the solution $u(x)$ and that $B_i, 1 = 1,2,...,m.$ are linearly independent, $P_m Ker\ D^m$ (see de Boor[2]).
With this assumption in mind, we can write every element
$f, f \in H^m, [a,b] = \{ f \in C^{m-1}\ \ [a,b; D^{m-1} f$ is absolutely continuous and $D^m f \in L_1[a,b]\},$
as,

$$f(x) = (P_0 f)(x) + \int_a^b G_0(x,t) D^m f(t) dt \tag{13}$$

where, $P_0 f$ is the element of P_m, which satisfies

$$B_i(P_0 f) = B_i f; i = 1,2,...,m,$$

and
$G_0(x,t)$ is the Green's function for the problem

$$\left.\begin{array}{c} D_m G_0 = \delta(x,t) \\ \text{and} \\ B_m G_0 = 0 \end{array}\right\} \tag{14}$$

Thus, equation (5.1) can be written in the form.

$$y = D^m u, x \in (a,b) \tag{15}$$

and such that

$$y = Ty \tag{16}$$

where the operator *T*, is defined by
$(Tf)(x) = (Fu)(x)$
$\quad = F(x, f^m(x), f^{(-m+1)}(x)..., f^{-1}(x)$
and

$$f_{-m+1}(x) - D_i(P^0 f)(x) + || \int_a^b G_1(x,t) f(t) dt, \tag{17}$$

with $1 = 0,1,2,...,m-1$

$$G_i(x,1) = \left(\frac{\partial}{\partial_x}\right)^1 G_0(x,1), i = 0,1,...,m-1 \tag{18}$$

(See de Boor [3]).

The equation (17) and (16) follow directly from the representation (13) above.

In particular, since $(P^0 f)(x) \in P_m,$ we integrate (17) *m* times and then differentiate *I* times. The result gives back equation (13).

As a result of the representation (15) which follows from (14) we find that the Green's Function has been used to transform the boundary value problem (12) into an equivalent operator equation (16),

i.e.

$$Y_N - D^m U_N, Y_{N\in} P_r \tag{19}$$

such that,

$$Y_N(X_{j,k}) = (T_{yN})(X_{j,k}) \tag{20}$$

Next, let P be the linear projection operator associated with polynomial interpolation at the r points, $P_1, P_2, \ldots, P_r$

Then, we can take $P\Delta$ as the projection operator associated with polynomial interpolation at the collocation points [4,5].

i.e., with the mapping P on $C[-1,1]$, there is associated collocation projector operator, $P\Delta$ and $C\Delta$, where

$$C\Delta = C(X_0, X_1)\,x\ C(X_1, X_2)\ \text{x}\ \ldots,\ \text{x}\ C(X_{N-1}, X_N)$$

and where,

$|\Delta| = max(X_{j=1}, X_j)$

we can therefore write equation (5, 11) as (21)

We now prove the following theorem to show that the iteration method defined by (21) is convergent.

Theorem (2): Suppose the collocation procedure defined by (21) is consistent such that for every *u.*

$\| y - P\Delta \text{ y} \| = 0$ and stable such that the discrete operators are uniformly invertible. And

$\| (P\Delta\ TP\Delta)^{-1} \| < C,$ a constant,

then, the method converges such that

$$\| y - P\Delta\, y \| \to 0$$

Proof: Let $P\Delta$ be the projection operator associated with the polynomial interpolation at the collocation points.

$P\Delta\ y = y$

Also,

$Ty = y$

$y = P\Delta\ y = P\Delta\ Ty = P\Delta\ TP\Delta\ y$

i.e.

$P\Delta\ \text{T}y = P\Delta\ y$

Hence,

$P\Delta\ \text{T}P\Delta\ y + P\Delta\ TP\Delta\ y = P\Delta\ y$

$$P\Delta\ \text{T}P\Delta\ y + P\Delta\ T(y - P\Delta\ y) = P\Delta\ y$$

Let

$$(P\Delta\ \text{T}\, P\Delta)^{-1} = \text{A}$$

Then,

$$y + AP\Delta\, T(y - P\Delta\, y) = AP\Delta\, y$$

Next, we put

$$AP\Delta\, y = y_n$$

$$y + AP\Delta\,\text{T}\ (y - P\Delta\, y) = Y_n$$

$$\| y - y_n \| = \|\ \ AP_\Delta T(y - P_\Delta y) \|$$

i.e.

$$|| y - y_n ||< c || T || y - P_\Delta ||\to 0$$

we observe that the projection equation

$$y = P_\Delta Ty \tag{23}$$

is similar to Newton's iteration formula for the solution of problem of the sum form. We therefore convert the equation (23) into an iterative method [4] thus:

Now,

$$y = Ty = 0 = y - P_\Delta y \tag{24.}$$

i.e.

$y = P_\Delta y$

$$y_{n+1} = P_\Delta (y_n) \tag{25.}$$

This completes the proof

4.0 **Numerical Computations**

1. Solution of the Differential Equation

$$y^n = 2(1 + 2x)y = 0;\quad Y(0) = 1, y(1) = 1$$

2.

x	y_c	y_a	$dy\ e/dx$	$dy\ a/dx$	$d^2\ y e/dx$	$d^2 y\ a/dx$
0	1	1	0	$3.9639\,E^{-05}$	2	2
1	1.01005	1.010054	0.20201	0.2020516	2.060502	2.060341
2	1.040811	1.040819	0.4163243	0.4163719	2.248151	2.248237
3	1.094174	1.094187	0.6565046	0.656564	2.582251	2.582406
4	1.173511	1.173528	0.9388087	0.9388884	3.098069	3.098329
5	1.284025	1.28405	1.284025	1.284139	3.952076	3,852497
6	1.433329	1.433365	1.719995	1.72016	4.930654	4.931268
7	1.632316	1.433365	1.719995	1.72016	4.930654	4.931268
8	1.896481	1.896558	3.03437	3.034603	8.647955	8.646956
9	2.247908	2.247997	4.046236	4.046062	11.77904	11.77006
10	2.718282	2.718281	5.436565	5.434301	16.3097	16.27087

Keys $Y_c = Y$ - exact, $Y_a = Y$ - approximate.

3. Solution of the Differential Equation $y^n + y = 0;\ Y(0), y^1(1) = 1$

x	y_c	y_a	$dy\ e/dx$	$dy\ a/dx$	$d^2\ y e/dx$	$d^2 y\ a/dx$
0	0	0	1.850816	1.850795	0	0
.1	0.1847733	0.1847712	1.841569	1.841569	-0.18847733	-0.1847683
.2	0.3677003	0.3676963	1.813923	1.813803	-0.3677003	-0.3676905
.3	0.5469534	0.5469475	1.768152	1.768133	-0.5469534	-0.5469691

.4	0.7207416	0.7207338	1.704714	1.704697	-0.7207416	-0.7206231
.5	0.8873283	0.887319	1.624244	1.624229	-0.8873283	-0.7206231
.6	1.045049	1.045038	1.527544	1.527532	-1.045049	-1.045024
.7	1.192328	1.192316	1.415582	1.415572	-1.192328	-1.1923
.8	1.327694	1.327681	1.289476	-1.289469	-1.327694	-1.327664
.9	1.449794	1.44978	1.150485	1.150482	-1.449794	-1.449761
.10	1.557408	1.55394	0.9999998	0.9999999	-1.557408	-1.557372

3. Solution of the Differential Equation $y^{n} = y^{n} / x;\ \ Y(1) = Y(2) = 0, Y'(1) = 1$

X	y_c	y_a	$dy\ \ e/dx$	$dy\ \ a/dx$	$d^2\ y\,e/dx$	$d^2y\ \ a/dx$
.1	-1.136373	-1.363273	1.363273	-0.1690358	-1.090909E-2	27.54681
.2	-1.000145	-0.976882	1.360727	1.042757	-4.363637E-2	4.492152
.3	-0.8643727	-0.8583734	1.353818	1.273311	-9.818183E-2	0.972177
.4	-0.7296	-0.7280434	1.340364	1.319219	-0.1745455	0.105405
.5	-0.5965909	-0.5961919	1.318182	1.312808	-0.2727273	-0.200277
.6	-0.4663273	-0.4662183	1.285091	1.283784	-0.3927273	-0.375712
.7	-0.340009	-0.3399733	1.238909	1.238564	-0.5345455	-0.531112
.8	-0.2190545	-0.2190418	1.177454	1.177329	-0.6981819	-0.697367
.9	-0.1050999	-0.105097	1.098545	1.098496	-0.8836365	-0.883115
1.0	1.192093E-7	1.192093E-7	0.9999999	0.9999999	-1.090909	0.090485

3. Solution of the Differential Equation $y^{n} = y^{n} / x;$ (Continued)

X	$d^2\ y\,e/dx$	$d^2y\ \ a/dx$
.1	-0.2181818	125.5954
.2	-0.4363636	14.0322
.3	-0.6545455	1.971961
.4	-0.8727273	-0.3702481
.5	-1.090909	-1.018406

.6	-1.309091	-1.311337
.7	-1.527273	-1.533202
.8	-1.745455	-1.74718
.9	-1.963637	-1.964277
1.0	-2.181818	-2.183546

Remarks: it is very clear from the tabulated results above that the function and its derivatives up to and including the highest-order, have the same order of accuracy. These remarkable results demonstrate the super convergence of the method.

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