

Model Simplification for Large Systems

O. Taiwo

Chemical Engineering Department,

Obafemi Awolowo University, Ile – Ife, Nigeria

Abstract

This work deals with methods for simplifying complex models of plants or controllers. Since it is impossible to deal in detail with all the currently available techniques, this work concentrates on those methods which have general applicability in terms of being usable for both single input single output and multivariable continuous and discrete systems. The methods are illustrated by examples.

1. Introduction

The purposes of model simplification include (a) Computational economy (b) Design of simplified control laws and (c) Synthesis and construction of controllers. The specific methods considered in this work include those which involve the following simplification criteria (1) matching of Taylor series coefficients such as Pade approximation moment matching and continued fraction, (2) matching of Markov parameters and optimal computation of matching points; (3) minimization of the function of error between the responses of the original and simplified models (4) truncation and residualization.

2. Moment Matching

Let S denote an asymptotically stable system with input $u(t)$ and output $y(t)$, characterized by the transfer function

$$G(s) = A_M(s)/B_N(s) \quad M < N \quad (2.1)$$

where the polynomial $B_N(s)$ has degree N which is large. It is desired to replace S^* having a transfer function $G^*(s)$ of denominator degree $n, n < N$. In moment matching the following criterion is used.

$$\int_0^{\infty} t^k [g(t) - g^*(t)] dt = 0, \quad k = 0, 1, 2, \dots, (m+n) \quad (2.2)$$

Where $G^*(s)$ has $m+n+1$ unknown parameters to be determined and g, g^* are respectively the inverse Laplace transforms of $G(s)$ and $G^*(s)$. In this definition, neither G nor G^* need be rational. To ensure that the moments of g and g^* are piecewise continuous in $[0, \infty]$ and of exponential order $0[\exp(\sigma t)], 1 \rightarrow \infty, \sigma < 0$. In (2.2) the error of approximation ($g - g^*$) is weighted by the increasing function t^k , hence it follows that the larger $(m+n)$, the more rapidly $(g - g^*)$ tends to zero as $t \rightarrow \infty$.

We denote

$$M_k(g) = \int_0^{\infty} t^k g(t) dt \quad (2.3)$$

$$G^k(s) = \frac{d^k G(s)}{ds^k} \quad (2.4)$$

Then

$$M_k(g) = (-1)^k G^k(0) \quad (2.5)$$

Eqn. (2.5) can be easily proved using the definition of $G(s)$ and directly differentiating under the integral sign. Let the function g be piecewise continuous on $[0, \infty]$ and of exponential order $0[\exp(\sigma t)], t \rightarrow \infty, \sigma < 0$ and let $G^*(s)$ be the

$(m + n)$ moment approximant of $G(s)$, then

$$M_k(g - g^*) = 0, k = 0, 1, 2, \dots, (m + n) \quad (2.6)$$

2.1 Computational Procedure For Getting Moment Approximants.

- (i) Given a complex model $G(s)$
- (ii) Postulate a simple model $G^*(s)$
- (iii) Compute $M_k(g) = M_k(g^*), k = 0, 1, \dots, (m + n)$
- (iv) Equate the moments i.e.

$$M_k(g) = M_k(g^*), k = 0, 1, \dots, (m + n)$$

hereby obtaining the unknown parameters of $G^*(s)$. We now consider examples of continuous and discrete system reduction.

2.2 Discrete Data System

Consider a discrete dynamical system characterised by the z-transfer function

$$G(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n} \quad (2.7)$$

$$= \sum_{i=0}^{\infty} g(i) z^i \quad (2.8)$$

where $g(iT)$ is the unit impulse response of the dynamic system with a constant sampling period T . The moments $\beta_k, k = 0, 1, 2, \dots$, for the system (2.15) are defined as

$$\beta_k = \sum_{i=0}^{\infty} (iT)^k g(iT) \quad (2.9)$$

The sampled system transfer function $G^+(s)$ is obtained by letting $z = \exp(Ts)$ in (2.15)

$$G(s) = \sum_{i=0}^{\infty} g(iT) \exp(-iT_s)$$

$$\sum_{i=0}^{\infty} g(iT) [1 - iT_s + \frac{1}{2!} (iT_s)^2 - \frac{1}{3!} (iT_s)^3 + \dots] \quad (2.10)$$

Hence

$$\beta_k = (-1)^k \frac{d^k G^*(s)}{ds^k} \Big|_{s=0} \quad (2.11)$$

Now, transforming the variable s to $lnz/T = 0$ so that

$$ds = dz/Tz$$

Equation (2.19) becomes

$$\beta_k = (-1)^k [(Tz d/dz)^k G(z)]|_{z=0} \quad (2.12)$$

In view of the limited usefulness of the direct application of (2.12) for model reduction, a simpler and less tedious method of equivalently obtaining reduced models for discrete systems will also be demonstrated later.

3. Relationship Between Moments, Continued Fraction And Pade Approximants (Zakian, 1973)

Definition

Let G be analytic at $s = 0$, then G^* is said to be the (m/n) Pade approximant of G , iff G is an (m/n) rational function and the leading $(m + n + 1)$ terms of the Maclaurin series of $G - G^*$ are zero.

$$\int_0^\infty t^k [g(t) - g^*(t)] dt = 0, k = 0, 1, 2, \dots, (m + n) \quad (2.2)$$

when g^* is an $(m+n)$ moment approximant of g . Consider the rational function $G^*(s)$ numerator degree m and denominator degree m . In most practical applications $n > m$.

Theorem 3.1.

Let the function g be piece wise continuous on $[0, \infty)$ and of exponential order $0[\exp(\sigma t)]$

$t \rightarrow \infty, \sigma < t$, and let $G^*(s)$ be (m/n) Pade approximant of $G(s)$, then

$$M_k(g - g^*) = 0, k = 0, 1, 2, \dots, (m + n) \quad (2.6)$$

iff $G^*(s)$ is a, s

Proof:

To prove sufficiency we note that the hypotheses imply that $G(s)$ and $G^*(s)$ are analytic at $s=0$, and hence the terms $G^{(k)}(0)$ and $G^{*(k)}(0)$, exist. Furthermore, $G^*(s)$ is the (m/n) Pade approximant of $G(s)$, hence

$$G^k(0) = G^{*(k)}(0), k = 0, 1, 2, \dots, (m + n) \quad (3.1)$$

Consequently, using (2.5) we get (2.6).

To prove necessity we suppose that $G^*(s)$ is not a.s., then $M_k(g^*)$ is not finite; thereby invalidating (2.6). Hence, it is necessary that $G^*(s)$ be a.s

Theorem 3.2

Let g be piecewise continuous on $[0, \infty)$ and of exponential order $O[\exp(\sigma t)]$, $t \rightarrow \infty, \sigma < 0$, let g^* be an $(m+n)$ moment approximant of g and let $G^*(s)$ be an (m/n) rational function. Consequently g^* is unique and G^* is the (m/n) Pade approximant of $G(s)$

Proof:

The assumptions imply that equation (2.6) are satisfied and that (2.5) holds for g and g^* consequently (3.1) must hold and hence since $G^*(s)$ is rational, it must be the (m/n) Pade approximant of $G(s)$. The uniqueness of g^* follows from the uniqueness of Pade approximation and the one-one property of the Laplace transform.

Theorem 3.3

Let g be piecewise continuous on $[0, \infty)$ and of exponential order $O[\exp(\sigma t)]$, $t \rightarrow \infty, \sigma < 0$, and let the Laplace transform $G^*(s)$ be an a.s, (m/n) rational function, then g^* is the $(m+n)$ moment approximant of g iff G^* is the (m/n) Pade approximant of G .

Proof:

Sufficiency is proved by Theorem 3.1 and necessity by Theorem 3.2

Continued Fraction:

Note that whenever continued fraction is mentioned without any qualification, the second Cuer form is meant

- (i) Just as for the Pade approximant, the continued fraction approximant may turn out to be unstable
- (ii) The continued fraction approximant G^* is the $(n-1)/n$ Pade approximant of G
- (iii) Condition (2.2) is satisfied iff G^* is a.s.

4. Pade Approximant

4.1 Continuous Systems

Definition 4.1

Consider a function

$$f(x) = G_0 + G_1x + G_2x^2 + \dots \tag{4.1}$$

and a rational $V_m(x)/T_n(x)$ where $V_m(x)$ and $T_n(x)$ are m th- and n th- order polynomials in x respectively. The rational function $V_m(\cdot)/T_n(\cdot)$ is said to be a Pade approximant or $f(x)$ iff the first $(m+n+1)$ terms of the power series expansion of $f(x)$ and $V_m(x)/T_n(x)$ are identical.

For the function $f(x)$ in (4.1) to be approximated, let the following Pade approximant be defined as

$$\frac{V_m(x)}{T_n(x)} = \frac{V_0 + V_1x + V_2x^2 + \dots + V_mx^m}{T_0 + T_1x + T_2x^2 + \dots + x^n} \quad (4.2)$$

For the first $(m+n+1)$ terms of (4.1) and 4.2) to be equivalent, it becomes apparent that the following set of relations must hold with the understanding that $G_k = 0(k < 0)$:

$$\sum_{i=0}^{n-1} G_{k-i}T_i = -G_{k-n} \quad (m+1 \leq K \leq m+n) \quad (4.3)$$

$$V_k = \sum_{i=0}^k G_{k-i}T_i \quad (0 \leq k \leq m) \quad (4.4)$$

Example 4.1

Find the 1/2 P.a. of

$$G(s) = \frac{1 - e^{-2s}}{S+1} \quad (4.7)$$

$$G^*(s) = \frac{V_0 + V_1s}{V_0 + T_1s + s^2} \quad (4.8)$$

To determine (4.8) one needs the first few terms of the Maclaurin expansion of (4.7)

$$G(s) = 0 + 2s - 4s^2 + 16/3s^3 - 6s^4 + 6.266667s^5 - \dots$$

Hence, from (4.5)

$$\begin{bmatrix} -4 & 2 \\ 16/3 & -4 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} \begin{bmatrix} 3/4 \\ 3/2 \end{bmatrix}$$

from (4.6)

$$\begin{bmatrix} V_0 \\ V_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3/4 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$$

or

$$G^* = \frac{3/2s + 0}{s^2 + 3/2s + 3/4} = \frac{3s + 0}{2s^2 + 3s + 3/2}$$

4.2 Discrete Systems

It has been established earlier that $s = 0$ is equivalent to $z = 1$. Hence in order to ensure similar steady state step responses for the original and reduced models, the Taylor series should be about $z = 1$, and the Taylor series sought is of the form

$$G(z) = \sum_{i=0}^{\infty} G_i(z-1)^i = \sum_{i=0}^{\infty} G_i W^i, \quad W = z-1 \quad (4.9)$$

One may then express the reduced model in the w-domain and later transform to the z-domain after simplification. The equations are then as given in (4.5) and (4.6) where the simplified model has the form.

$$G^*(w) = \frac{V_0 + V_1 w + \dots + V_m w^m}{T_0 + T_1 w + \dots + w^n} \quad (4.10)$$

5. Continued Fraction Expansions

Consider a SISO system with a transfer function representation

$$G(s) = \frac{a_{2n}s^{n-1} + \dots + a_{23}s^2 + a_{22}s + a_{21}}{a_{1,n+1}s^n + \dots + a_{13}s^2 + a_{12}s + a_{11}} \quad (5.1)$$

out of the several continued fraction forms into which (5.1) can be expanded there are three basic forms of particular interest to system engineering. These are called the Cauer Forms. The first Cauer form

$$G(s) = \frac{1}{h_1 s + \frac{1}{h_2 + \frac{1}{h_3 s + \frac{1}{h_4 s + \frac{1}{h_5 + \dots}}}}} \quad (5.2)$$

has $2n$ terms in h_i and represent a Maclaurin series expansion around $s = \infty$

The second Cauer form

$$G(s) = \frac{1}{k_1 + \frac{1}{k_{2/s} + \frac{1}{k_3 + \frac{1}{k_{4/s} + \dots}}}}} \quad (5.3)$$

Has $2n$ terms in k_i and represents a Maclaurin series about $s = 0$.

The third Cauer form

$$G(s) = \frac{1}{d_1 + f_1 s + \frac{1}{d_1/s + f_2 + \frac{1}{d_3 + f_3 s + \frac{1}{d_4/s + f_4 + \dots}}}}} \quad (5.4)$$

Has n terms each in d_i and T_i and is equivalent to a Maclaurin series expansion about $s = \infty$.

$$\dots$$

$$a_{n,1} \quad a_{n,2} \quad \dots \quad f_n = a_{n,2}/a_{n+1,1} \quad (5.7)$$

$$d_n = a_{n,1}/a_{n+1,1}a_{n+1,1}$$

where $d_p = a_{p1}/a_{p+11}$ $p = 1, 2, 3, \dots, n$; $f_p = a_{pn+2-p}/a_{p+1,n+1-p}$
 $d_{j,e} = a_{j-2,e;1} - d_{j-2}a_{j-1,e;1} - f_{j-2}a_{j-1,e}; j = 3, 4, \dots, n+1$; $e = 1, 2, 3, \dots$

5.2 Discrete Data Systems

First Cuer form This is equivalent to power series expansion about $z = \infty$. It is instructive in this case to write the original transfer function as

$$\frac{b_{21}z^{n-1} + b_{22}z^{n-2} + \dots + b_{2n}}{b_{11}z^n + b_{12}z^{n-1} + \dots + b_{1,n+1}}$$

so that the Routh array takes the form

$$\begin{array}{ccccccc} b_{11} & b_{12} & b_{13} & \dots & b_{1,n+1} & & \\ b_{21} & b_{22} & b_{23} & \dots & b_{2,n} & & \\ b_{31} & b_{32} & \dots & & & & \\ b_{41} & b_{42} & \dots & & & & \end{array}$$

where $b_{j,i} = b_{j-2,i+1} = h_{j-2}b_{j-1,j+1}$; $j = 3, 4, \dots, 2n+1$; $i = 1, 2, \dots, n$;
 $h_p = b_{p,1}/b_{p+1,1}$

Example 5.5

Expand the z transfer function

$$G(z) = \frac{2z + 1}{z^2 + 2z + 1} \quad (5.8)$$

into the first Cuer form. The Routh Array is

$$\begin{array}{cccc} 1 & 2 & 1 & \\ 2 & 1 & & \\ 3/2 & 1 & & \text{hence } h_1 = 1/2, h_2 = 4/3; h_3 = -9/2; h_4 = -1/3 \\ -1/3 & & & \\ 1 & & & \end{array}$$

where $b_{31} = b_{12} - (1/2)(1) = 3/2$

$$b_{32} = b_{13} - 1/2(b_{23}) \Rightarrow 1 - (0) = 1$$

$$b_{41} = b_{22} - (4/3)(b_{32}) \Rightarrow 1 - 4/3(1) = -1/3$$

$$b_{42} = b_{23} - 4/3(b_{33}) = 0$$

$$b_{51} = b_{32} + (9/2)(b_{42}) = 1 - 0 = 1.$$

$$G(z) = (z/2 + (4/3 + (-9z/2 + (-1/3)^{-1})^{-1})^{-1})^{-1}$$

$$= \frac{2z+1}{z^2+2z+1}$$

Demonstrating that $2n$ quotients are needed to construct $G(z)$ of order n . In order to obtain a reduced first order model one just inverts the continued fraction involving the first two quotients h_1 and h_2 .

As stated earlier the second and third Cuer forms are more useful in obtaining, reduced models as they preserve the steady state characteristics of the original model. Noting that $z=e^{Ts}$, expansion about $s = 0$ is equivalent to expansion about $z = 1$. Hence given the transfer function

$$G(z) = \frac{a_{21} + a_{22}z + \dots + a_{2n}z^{n-1}}{a_{11} + a_{12}z + \dots + a_{1,n+1}z^n}$$

transform to the w plane where $z - 1 = w$ to have

$$G(w) = \frac{b_{21} + b_{22}w, \dots + b_{2n}w^{n-1}}{b_{11} + b_{12}w + \dots + b_{1,n+1}w^n}$$

Then the continued fraction expansion in the second Cuer form takes the form (5.3)

with w replacing s . Similarly for the third Cuer form (5.4) except that s is also replaced by w . The truncated continued fraction would then have to be transformed back to the z -domain.

6. Matrix Pade Method

Suppose the system has a square transfer function $G(s)$ which possesses a power series expansion

$\left(\sum_{j=0}^{\infty} G_j s^j \right)$ about $s = 0$. The reduction procedure consists of approximating $G(s)$ by a rational matrix function expressed as a quotient of polynomial matrices in either the 'left' or 'right' form,

$$L(s) = P^{-1}(s)Q(s)$$

$$R(s) = V(s)T^{-1}(s)$$

respectively. The polynomial matrices will have forms

$$T(s) = \sum_{k=0}^n T_k S_k \quad (T_n = 1)$$

$$T(s) = \sum_{k=0}^m T_k S_k \quad (m < n)$$

for $R(s)$ and similarly for $L(s)$. It follows that both $L(s)$ and $R(s)$ vanish asymptotically as $s \rightarrow \infty$.

6.1 Matrix Pade Approximants

For given values of m and n the rational matrices $L(s)$ and $R(s)$ are called respectively, left and right matrix (m/n) Pade approximants

$$L(s) = G(s) + 0 \quad (s^{m+n+1})$$

$$R(s) = G(s) + 0 \quad (s^{m+n+1})$$

where the notation means that the power series expansions on both sides exist and agree up to terms of degree $(m + n)$

Theorem 19.2.1:

Under the above conditions

$$L(s) = R(s)$$

Proof:

See (Shamash 1976, Cook 1981 and Taiwo, 1995).

6.2 Construction of Reduced Models

In view of Theorem 1, we consider only $R(s)$. If this is to be a matrix m/n Pade approximant to $G(s)$.

$$V(s) = G(s) - T(s) + 0(s^{m+n+1})$$

The equations for $T(s)$ and $V(s)$ then

$$\sum_{i=0}^{n-1} G_{k-i} T_i = G_{k=m} \quad (m+1 \leq k \leq m+n)$$

$$V_k = \sum_{i=0}^k G_{k-i} T_i \quad (0 \leq k \leq m)$$

where $G_k = 0$. In full form, these equations take the forms (4.5) and (4.6)

Theorem 8.2.1:

If the above equations for the coefficients of $T(s)$ have a solution, such that all the zeros of $\det T(s)$ lie in $R(s) < 0$, then the resulting $R(s)$ will have the same impulse response matrix moments as $G(s)$ of orders 0 to $(m+n)$.

Proof:

(See Cook 1981 and Taiwo 1995)

6.3 Discrete Data Systems:

In this case the high order system is $G(z)$. The Taylor series is about $s = 0$ or $z = 1$. The reduced model takes the form $R(w) = V(w)T^{-1}(w) \quad z - 1 = w$. Having obtained $R(w)$, obtain $R(z)$ by substituting $w = z - 1$ in $R(w)$. This 1st operation could be cheaply done using synthetic division (Taiwo, 1989).

6.4 Matrix Continued Fraction

6.4.1 Continuous Systems

Given a high order system $T(s)$. It can be expressed in tree Cauer forms. The first matrix Cauer form is

$$T(s) = [H'_1 s +] H'_2 + [H'_3 s + [H'_4 + [\dots]^{-1}[-1]^{-1}]^{-1}]^{-1}$$

The second matrix Cauer form is

$$T(s) = [H'_1 +] H'_2 s^{-1} + [H' + [H'_4 s^{-1} + [\dots]^{-1}[-1]^{-1}]^{-1}]^{-1}$$

and the third matrix Cauer form is

$$T(s) = [k_1 + k'_1 s + k'_2 s^{-1} + k'_2 + [k_3 + k'_3 s + [\dots]^{-1}[-1]^{-1}]^{-1}]^{-1}$$

The high order system has the form

$$\begin{aligned} T(s) &= [A_{2,n}s^{n-1} + A_{2,n-1}s^{n-2} + \dots + A_{22}s + A_{21}] \times [A_{1,n=1}s^n + A_{1,n}s^{n-1} + \dots + A_{12}s + A_{11}]^{-1} \\ &= [B_{21}s^{n-1} + B_{22}s^{n-2} + \dots + B_{2,n-1}s + B_{2,n}] \times [B_{11}s^n + B_{12}s^{n-1} + \dots + B_{1,n}s + B_{1,n+1}]^{-1} \end{aligned}$$

The H_{is} , H_{is} , k_i and $'k_i$ are constant matrix quotients obtained from the matrix Routh array and the generalized algorithm s follows:

$$\begin{aligned} H_1 &= \begin{matrix} B_{11} & B_{12} & B_{13} & \dots & B_{1,n+1} \\ B_{21} & B_{22} & B_{23} & \dots & B_{2,n} \\ B_{31} & B_{32} & & & \\ B_{41} & B_{42} & & & \\ \dots & \dots & & & \\ \dots & \dots & & & \\ B_{2n+1,1} & & & & \\ A_{11} & A_{12} & A_{13} & \dots & A_{1,n+1} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2,n} \\ A_{32} & & & & \\ A_{41} & A_{42} & & & \\ \dots & \dots & & & \\ A_{2n+1,1} & & & & \end{matrix} A_{ii} A_{21}^{-1} \\ H_2 &= \begin{matrix} A_{21} A_{31}^{-1} A_{31} \\ A_{32} \\ A_{41} & A_{42} \\ \dots & \dots \\ A_{2n+1,1} \end{matrix} \end{aligned}$$

$$\begin{array}{rcc}
& A_{11} & A_{12} & A_{13} & \dots & A_{1,n+1} \\
K_1 = & A_{11}A_{21}^{-1} & & & & K_1 = A_{1,n+1}A_{2n-1} \\
& A_{21} & A_{22} & A_{23} & \dots & K_{2n} \\
K_2 = & A_{21}A_{31}^{-1}A_{31} & A_{32} & & & K_2 = A_{2,n}A_{3,n-1} \\
& A_{41} & A_{42} & & & \\
& \dots & \dots & & & \\
& \dots & \dots & & & \\
& A_{2n+1,1} & & & &
\end{array}$$

The elements of the third, fourth and subsequent rows in the Routh array are evaluated, respectively for the three Cauer forms by the following recursion:

$$\begin{aligned}
H'_p &= B_{p,1}B_{p+1,1}^{-1} & p &= 1, 2, 3, \dots, 2n \\
B_{j,e} &= B_{j-2,e+1} = H_{j-2}B_{j-1,c+1} & j &= 3, 4, \dots, 2n+1 \quad e = 1, 2, \dots, n \\
&\det B_{p+1,1} \neq 0 \\
H_p &= A_{p,1}A_{p+1,1}^{-1}, & p &= 1, 2, 3, \dots, 2n \\
A_{j,e} &= A_{j-2,e+1} - H_{j-2}A_{j-1,e+1} & j &= 3, 4, \dots, 2n+1 \quad e = 1, 2, \dots, n \\
&\det A_{p+1,1} \neq 0 \\
k_p &= A_{p,1}A_{p+1,1}^{-1}, & p &= 1, 2, 3, \dots, n \\
k_p &= A_{p,n+2-p}A_{p+1,n+1-p}^{-1}, & j &= 3, 4, \dots, n+1 \\
A_{j,e} &= A_{j-2,e+1} - k_{j-2}A_{j-1,e+1} - K_{j-2}A_{j-1,e} & e &= 1, 2, \dots
\end{aligned}$$

6.4.2. Discrete Data System.

For the Cauer first form, no transformation is necessary on $T(z)$ before computation of the matrix quotients. For the second and third forms, first transform $T(z)$ to the w plane where $z = w + 1$. Then compute the quotients of $T(w)$. If a reduced model is desired truncate the continued fraction appropriately and obtain the reduced model $R(z)$ by substituting $w = z - 1$ in $R(w)$.

7. Optimal Model Reduction For Systems With Time Delay

7.1 Problem Formulation

Consider a stable linear time-invariant system with a time delay, whose transfer function is given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{(a_0 + \dots + a_{n-1}s^{n-1}) \exp(-ds)}{1 + \dots + b_{n-1}s^{n-1} + b^n s^n} \quad (7.1)$$

where $U(s)$ and $Y(s)$ are the Laplace transforms of the input variable $u(t)$ and the output response $y(s)$, respectively. The objective is to compute an m th order ($m < n$) stable reduced order model with a time delay.

$$G_r(s) = \frac{Y_r(s)}{U(s)} = \frac{(c_0 + \dots + c_{m-1}s^{m-1}) \exp(-\tau s)}{1 + \dots + d_{m-1}s^{m-1} + d_m s^m} \quad (7.2)$$

such that for a suitable input $U(s)$, the integral of squared error (ISE)

$$J = \int_0^{\infty} (y(t) - y_r(t))^2 dt \quad (7.3)$$

is minimized. For the purposes of this work,

$$U(s) = \frac{U_0}{S} + u_1 + \frac{u_2}{u_3 s + 1} \quad (7.4)$$

where the u_i are constants. The problem then is to find the coefficient c_1 and d_1 and

the time delay τ in (7.2) such that (7.3) is minimized subject to the constraints that all the poles p_i of $G_i(s)$ lie in the open left half plane,

$$\text{Re}(p_i) < 0, \quad i = 1, 2, \dots, m \quad (7.5)$$

with zero steady state error in the output responses, that is,

$$e(t) = y(t) - y_r(t) = 0 \quad \text{for } t \rightarrow \infty \quad (7.6a)$$

where $\text{Re}(p_i)$ denotes the real part of the complex number p_r . The steady state constraint (7.6a). The last constraint is

$$c_0 = a_0 \quad (7.6b)$$

whenever $u_0 \neq 0$ in (7.4), thereby ensuring the existence of the indefinite integral (7.3). The last constraint is

$$\tau \geq 0 \quad (7.7)$$

This is therefore a parameter optimization problem which can be efficiently solved

by a suitable optimization procedure provided that the stability constraints (7.5) are satisfied during the course of the optimization and that (7.3) can be cheaply computed. In order to cheaply compute (7.3) one makes use of Parseval's identity.

$$J = \int_0^{\infty} c^3(t) dt = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} E(s)E(-s), i = \sqrt{-1} \quad (7.8)$$

where

$$E(s) - Y(s) - Y_r(s) = [G(s) - G_t(s)]U(s) \quad (7.9)$$

As it will be shown in the next section, the frequency domain expression of ISE in (7.8) cannot be directly evaluated using the recursive algorithm given by Astrom (1970). The introduction of a simple reformulation overcomes this problem. In order to ensure sufficient accuracy, this algorithm has been coded in MATLAB in this work. This also ensures that the floating under- and over-flow limits are not easily violated in general, capital letters denote Laplace transform variables.

7.2 Computation of ISE

$E(s)$ in (9) can be decomposed into its steady state and transient components.

$$E(s) = [(m/s + Z(s))c^{-ds} = m/s + Z_\pi(s)c^{ts}] \quad (7.10)$$

where

$$m = a_0 u_0 \quad (7.11)$$

In the time domain (7.10) becomes

$$e(f) = (m^* 1(t-d) + * 1(t-d)) - (m^* 1(t-\tau) + z_r(t-\tau)^* 1(t-\tau)) \quad (7.12)$$

where $*$ represents ordinary multiplication and the delayed unit step function is given by

$$1(t-d) = \begin{cases} 0, & t < d \\ 1, & t \geq d \end{cases} \quad (7.13)$$

in terms of (7.12), j which is a function of c_i, d_i , and τ in (7.2) can be written as

$$J = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \quad (7.14a)$$

where

$$J_1 = \int_0^\infty [m^* 1(t-d) - m^* 1(t-\tau)]^2 dt \quad (7.14b)$$

which (without loss of generality) for $\tau > d$, becomes

$$J_1 = \int_d^\tau [m^* 1(t-d) - m^* 1(t-\tau)]^2 dt = m^{2*}(\tau - d) \quad (7.14c)$$

and for $\tau < d$,

$$J_1 = \int_{\tau}^d [m^* 1(t - \tau) - m^* 1(t - d)]^2 dt = m^{2*}(d - \tau) \quad (7.14d)$$

$$J_2 = \int_0^{\infty} z^2(t = d)^* 1(t - d) dt \quad (7.15a)$$

Which on application of Parseval's identity becomes

$$J_2 = \frac{1}{2\pi i} \int_{i\infty}^{i\infty} Z(s)Z(-s)ds \quad (7.15b)$$

while

$$J_3 = \int_0^{\infty} Z_r^2(t - \tau)^* 1(t - \tau) dt = \frac{1}{2\pi i} \int_{i\infty}^{i\infty} Z_r(s)Z_r(-s)ds \quad (7.16)$$

on application of Parseval's identity,

$$J_4 = -2 \int_0^{\infty} Z(t - d)^* 1(t - d)Z_r(t - \tau)^*(t - \tau) dt \quad (7.17a)$$

For J_4 , and equivalent expression in the frequency domain is obtainable after a simple reformulation of (7.17a). Suppose $\tau > d$, then as shown in the next section it is possible to reformulate J_4 to have the form

$$J_4 = -2 \int_0^{\infty} W(t - \tau)^* 1(t - \tau)Z_r(t - \tau)^* 1(t - \tau) dt \quad (7.17b)$$

where the form of the function w depends on the original function z . Note that the transformation from z to w is to ensure that all the terms under the integral have the larger delay. When $d > \tau$, the reformulation takes the form

$$J_4 = -2 \int_0^{\infty} Z(t - d)^* 1(t - d)W_r(t - d)^* 1(t - d) dt \quad (7.17c)$$

where again the transformation from z_r to w_r ensures that all the terms under the integral have the larger delay. Application of the Parseval's identity to (7.17b) and (7.17c) gives respectively

$$J_4 = \frac{-1}{\pi i} \int_{i\infty}^{i\infty} W(s)Z_r(-s)ds, \tau > d \quad (7.17d)$$

$$J_4 = \frac{-1}{\pi i} \int_{-i\infty}^{i\infty} Z(s)W_r(-s)ds, d > \tau \quad (7.17e)$$

For $\tau > d$, J_5 and J_6 take the forms

$$J_5 = 2 \int_0^{\infty} m^* 1(t-d) Z(t-d)^* 1(t-d) dt \quad (7.18a)$$

which, using the final value theorem of Laplace transform gives

$$J_5 = 2mZ(0) \quad (7.18b)$$

Similarly

$$J_6 = -2 \int_0^{\infty} m^*(t-\tau) Z(t-d)^* 1(t-d) dt \quad (7.19a)$$

which after the reformulation alluded to in (7.17b) gives

$$J_6 = -2 \int_0^{\infty} m^*(t-\tau) Z(t-\tau)^* 1(t-\tau) dt \quad (7.19b)$$

From the final value theorem of Laplace transform, (7.19b) is easily shown to be given by

$$J_6 = -2mW(0) \quad (7.19c)$$

For $d > \tau$, J_5 and J_6 respectively become

$$J_5 = 2 \int_0^{\infty} m^* 1(t-\tau) Z_{\tau}(t-\tau) \quad = 2mZ_r(0) \quad (7.20a)$$

$$\begin{aligned} J_6 &= -2 \int_0^{\infty} m^* 1(t-d) Z_{\tau}(t-\tau)^* 1(t-\tau) dt \quad = -2 \int_0^{\infty} m^*(t-d) W_{\tau}(t-d)^* 1(t-d) dt \\ &= -2m W_{\tau}(0) \end{aligned} \quad (7.21)$$

From the foregoing notice that J_2, J_3 and J_4 can now be easily evaluated using

Astrom's (1970) recursive algorithm, suitably programmed, while the remaining J_i are easily determined as indicated in the respective formulas. The minimization of J then facilitates the computation of the optimal $G_{\tau}(s)$.

8. Truncation, Residualization and optimal Hankel norm approximation (Skogestad & Postlethwaite, 1996)

Let (A, B, C, D) be a minimal realization of a stable system $G(s)$, and partition the

state vector x , of dimension n , into $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where x_2 is the vector of $n-k$ states

which we wish to remove. With appropriate partitioning of A, B , and C , the state-space equations become

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u \quad (8.1a)$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u \quad (8.1b)$$

$$Y = G_1x_1 + G_2x_2 + Du \quad (8.1c)$$

Truncation

A k -th order truncation of the realization $G = (A, B, C, D)$ is given by $G_a = (A_{11}, B_1, C_1, D)$. The truncated model G_a is equal to G at infinite frequency, $G(\infty) = G_a(\infty) = D$, but apart from this there is little that can be said in the general case about the relationship between G and G_a . If however, A is in Jordan form then it is easy to order the states so that x_2 corresponds to high frequency or fast modes. This is discussed next.

8.1.1 Model truncation

For simplicity, assume that A has been diagonalized so that

$$A = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1^T \\ b_2^T \\ \dots \\ b_n^T \end{bmatrix}, \quad C = [c_1 c_2 \dots c_n] \quad (8.2)$$

Then, if the λ_i are ordered so that $|\lambda_1| < |\lambda_2| < \dots$, the fastest modes are removed from the model after truncation. The difference between G and G_a following a k -th order model truncation is given by

$$G - G_a = \sum_{i=k+1}^n \frac{c_i b_i^T}{s - \lambda_i} \quad (8.3)$$

And therefore

$$|G - G_a|_\infty \leq \sum_{i=k+1}^n \frac{\sigma(c_i b_i^T)}{|\operatorname{Re}(\lambda_i)|} \quad (8.4)$$

It is interesting to note that the error depends on the residues $c_i b_i^T$ as well as the λ_i . The distance of λ_i from the imaginary axis is therefore not a reliable indicator of whether the associated mode should be included in the reduced order model or not.

An advantage of the model truncation is that the poles of the truncated model are a subset of the poles of the original model and therefore retain any physical interpretation they might have.

8.2 Residualization

In truncation, we discard all the states and dynamics associated with x_2 . Suppose that instead of this we simply set $x_2 = 0$, i.e. we residualize x_2 , in the state-space

equations. One can then solve for x_2 in terms of x_1 and u , and back substitution of x_2 , then gives

$$\dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \quad (8.5)$$

$$y = (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u \quad (8.6)$$

Let us assume A_{22} is invertible and define

$$A_r \triangleq A_{11} - A_{12}A_{22}^{-1}A_{21} \quad (8.7)$$

$$B_r \triangleq B_1 - A_{12}A_{22}^{-1}B_2 \quad (8.8)$$

$$C_r \triangleq C_1 - C_2A_{22}^{-1}A_{21} \quad (8.9)$$

$$D_r \triangleq D - C_2A_{22}^{-1}B_2 \quad (8.10)$$

The reduced order model $G_a(s)(A_r, B_r, C_r, D_r)$ is called a residualization of $G(s) = (A, B, C, D)$. Usually (A, B, C, D) will have been put into Jordan form, with the eigenvalues ordered so that x_2 contains the fast modes. Model reduction by residualization is then equivalent to singular perturbational approximation, where the derivatives of the fastest states are allowed to approach zero with some parameter ϵ . An important property of residualization is that it preserves the steady-state gain of the system, $G_a(s) = G(s)$. This is to be contrasted with truncation, which retains the system behaviour at infinite frequency. This contrast between truncation and residualization follows from the simple bilinear relationship $S \rightarrow \frac{1}{S}$ which relates the two (e.g. Liu and Anderson, 1989)

It is clear from the discussion above that truncation is to be preferred when accuracy is required at high frequencies, whereas residualization is better for low frequency modelling. Both methods depend to a large extent on the original realization and we have suggested the use of the Jordan form. A better realization, with many useful properties is the balanced realization which will be considered next.

8.3 Balanced Realizations

In words only: A balanced realization is an asymptotically stable minimal realization in which the controllability and observability Gramians are equal and diagonal. More formally; let (A, B, C, D) be a minimal realization of a stable, rational transfer function $G(s)$, then (A, B, C, D) is called balanced if the solutions to the following Lyapunov equations

$$AP + PA^T + BB^T = 0 \quad (8.11)$$

$$A^TQ + Q + C^TC = 0 \quad (8.12)$$

Are $P = Q = \text{diag} (\sigma_1, \sigma_2, \dots, \sigma_n) = \Sigma$, where $\sigma_1 \geq \sigma_2 \geq \dots \sigma_n > 0$, P and Q are

the controllability and observability Gramians, also defined by

$$P \triangleq \int_0^{\infty} e^{At} B B^T e^{Tt} dt \quad (8.13)$$

$$Q \triangleq \int_0^{\infty} e^{A^t} C^T C e^{At} dt \quad (8.14)$$

Σ is therefore simply referred to as the Grammian of $G(s)$. The σ_i are the ordered Hankel singular values of $G(s)$, more generally defined as $\sigma_i \triangleq \lambda_i^{1/2}(PQ), i = 1, \dots, n$. Notice that $\sigma_i = \|G\|_H$, the Hankel norm of $G(s)$.

Any minimal realization of a stable transfer function can be balanced by a simple state similarity transformation, and routines for doing this are available in MATLAB. For further details on computing balanced realizations, see Laub et al. (1987). Note that balancing does not depend on D .

So what is so special about a balanced realization? In a balanced realization the value of each σ_i is associated with a state x_i of the balanced system. And the size of σ_i is a relative measure of the contribution that x_i makes to the input-output behaviour of the system. Therefore if $\sigma_i \gg \sigma_2$, then the state x_i affects the input-output behaviour much more than x_2 , or indeed any other state because of the ordering of the σ_i . After balancing a system, each state is just as controllable as it is observable, and a measure of a state's joint observability and controllability is given by its associated Hankel singular value. This property is fundamental to balanced model reduction and Hankel norm methods which work by removing states having little effect on the system's input-output behaviour.

8.4 Balanced truncation and balanced residualization

Let the balanced realization (A, B, C, D) of $G(s)$ and the corresponding Σ be partitioned compatibly as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = [C_1 \quad C_2] \quad (8.15)$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \quad (8.16)$$

Where $\Sigma_1 = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n), \Sigma_2 = \text{diag}(\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_n)$ and $\sigma_k > \sigma$.

8.4.1 Balanced truncation

The reduced order model given by (A_{11}, B_1, C_1, D) is called a balanced truncation of the full order system $G(s)$. This idea of balancing the system and then discarding the states corresponding to small Hankel singular values was first introduced by Moore (1981). A balanced truncation is also a balanced realization (Pernebo and Silverman, 1982), and the infinity norm of the error between $G(s)$ and the reduced

order system is bounded by twice the sum of the last $n - k$ Hankel singular values, i.e. twice the trace of Σ_2 or simply "twice the sum of the tail" (Glover, 1984, Enns, 1984). For the case of repeated Hankel singular values, Glover (1984) shows that each repeated Hankel singular value is to be counted only once in calculating the sum

A precise statement of the bound on the approximation error is given in theorem 1 below.

Useful algorithms that compute balanced truncations without first computing a balanced realization have been developed by Tombs and Postlethwaite (1987) and Safonov and Chiang (1989). These still require the computation of the observability and controllability Gramians, which can be a problem if the system to be reduced is of very high order. In such cases the technique of Jaimoukha et al. (1992), based on computing approximate solutions to Lyapunov equations, is recommended.

8.4.2 Balanced residualization

In balanced truncation above, we discarded the least controllable and observable states corresponding to Σ_2 . In balanced realization, we simply set to zero the derivatives of all these states. The method was introduced by Fernando and Nicholson (1982) who called it a singular perturbational approximation of a balanced system. The resulting balanced residualization of $G(s)$ is (A, B_r, C_r, D_r) as given by the formula (8.7) to (8.10).

Liu and Anderson (1989) have shown that balanced residualization enjoys the same error bound as balanced truncation. An alternative derivation of the error bound, more in the style of Glover (1984), is given by Sama et al. (1995). A precise statement of the error bound is given in the following theorem.

Theorem 8.1

Let $(G(s))$ be a stable rational transfer function with Hankel singular values $\sigma_1 > \sigma_2 > \dots > \sigma_N$ where each σ_i has multiplicity r_i and let $G_a^k(s)$ be obtained by truncating or residualizing the balanced realization of $G(s)$ to the first $(r_1 + r_2 + \dots + r_k)$ states. Then

$$\|G(s) - G_a^k(s)\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \dots + \sigma_N) \quad (8.17)$$

The following two exercises are to emphasize that (i) balanced truncation preserves the steady state-gain of the system and (ii) balanced residualization is related to balanced truncation by the bilinear transformation $s \rightarrow s^{-1}$.

8.5 Optimal Hankel norm approximation

In this approach to model reduction, the problem that is directly addressed is the following given a stable model $G(s)$ of order (McMillan degree) n , find a reduced order model $G_h^k(s)$ of degree k such that the Hankel norm of the approximation error, $\|G(s) - G_h^k(s)\|_H$, is minimized.

The Hankel norm of any stable transfer function $E(s)$ is defined as

$$\|E(s)\|_n \triangleq p^{1/2}(PQ) \quad (8.18)$$

Where P and Q are the controllability and observability Gramians of $E(s)$. It is also the maximum Hankel singular value of $E(s)$. So in the optimization we seek an error which is in some sense closet to being completely unobservable and completely uncontrollable, which seems sensible.

The Hankel norm approximation problem has been considered by many but especially Glover (1984). In Glover (1984) a complete treatment of the problem is given, including a closed-form optimal solution and a bound on the infinity norm of the approximation error. The infinity norm bound is of particular interest because it is better than that for balanced truncation and residualization.

The theorem below gives a particular construction for optimal Hankel norm approximations of square stable transfer functions.

Theorem 8.2

Let $G(s)$ be a stable, square, transfer function $G(s)$ with Hankel singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k \geq \sigma_{k+1} \geq \sigma_{k+2} - \dots \sigma_{k+1} > \sigma_{k+1+1} \geq \dots \geq \sigma_n > 0$, then an optimal Hankel norm approximation of order k , $G_h^k(s)$, can be constructed as follows.

Let (A, B, C, D) be a balanced realization of $G(s)$ with the Hankel singular values reordered so that the Grammian matrix is

$$\begin{aligned} \Sigma &= \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1+1}, \dots, \sigma_n, \sigma_{k+1}, \dots, \sigma_{k+1}) \\ &\triangleq \text{diag}(\sum_1, \sigma_{k+1} 1) \end{aligned} \quad (8.19)$$

Partition (A, B, C, D) to conform with Σ

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = [C_1 \quad C_2] \quad (8.20)$$

Define (A, B, C, D)

$$A \triangleq \Gamma^{-1}(\sigma_{k+1}^2 A_{11}^T + \sum_1 A_{11} \sum_1 -\sigma_{k+1} C_1^T U B_1^T) \quad (8.21)$$

$$B = \Gamma^{-1}(\sum_1 B_1 + \sigma_{k+1} C_1^T U) \quad (8.22)$$

$$C = C_1 \sum_1 + \sigma_{k+1} U B_1^T) \quad (8.23)$$

$$D = D - \sigma_{k+1} U \quad (8.24)$$

where U is a unitary matrix satisfying

$$B_2 = -C_2^T U \quad (8.25)$$

And

$$\Gamma \triangleq \sum_1^2 -\sigma_{k+1}^2 \mathbf{1} \quad (8.26)$$

The matrix A has k "stable" eigenvalues (in the open left-half plane); the remaining ones are in the open right-half plane. Then

$$G_h^k(s) + F(s) \triangleq \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \quad (8.27)$$

Where $G_h^k(s)$ is a stable optimal Hankel norm approximation of order k , and $F(s)$ is an anti-stable (all poles in the open right-half plane) transfer function of order $n - k - l$. The Hankel norm of the error between G and the optimal approximation G_h^k is equal to the $(k + 1)$ 'th Hankel singular value of G .

$$\|G - G_h^k\|_H = \sigma_{k+1}(G) \quad (8.28)$$

9. Comparison/Summary of Methods/Further Work

Non-rational models

The moment matching and optimal methods are applicable to non-rational models in both directions in that a complex model with or without non-rational terms such as time delays may be reduced to models with or without time delays. Pade method can reduce a complex model with non-rational terms to a model without time delays. The continued fraction method is inapplicable to non-rational models. By definition a moment approximant and an optimal approximant are only defined for an originally complex stable model if and only if the reduced models are also stable. Continued fraction and Pade approximants of stable models may be unstable

Multivariable Models

All the methods, suitably formulated are applicable to multivariable models. When Pade and continued fraction techniques are used the reduced model is of order $n_0 n = n_i$, is the number of inputs/outputs of a square system and n is the integer referred to in the m/n approximant. In the case of continued fraction in either the first or second Cuer form, n is half the number of quotients used for the reduction.

For the third Cuer form, n is the number of high-or low-frequency quotients used in the case of Routh and stability equation, the order (Rosenbrock 1970, Taiwo 1993) of the reduced model is nd^*nm , where $nm = \min\{n_0, n\}$ and nd is the degree of the common denominator of the reduced model. The order of the reduced model obtained using Modal/Pade, Routh-Pade and Stability equation-Pade methods is also equal to the above expression. However, Taiwo (1995) has proposed a method based on Modal-Pade technique which minimizes the reduced model order.

Stability of the Reduced Model

Moment optimal Routh, Stability Equation, Modal, Aggregation and Perturbation methods produce stable reduced models for originally stable models. Reduced models obtained using Pade and continued fraction methods may turn out to be unstable for originally stable models. For these latter methods it is possible to still obtain stable and accurate methods by using expansion not only about $s = 0$ but about many other points (Taiwo 1995)

Optimal Approximants

Active research is still continuing on model reduction. The weighted Hankel norm and balanced realization techniques (Moore 1981, Glover 1984, Enns 1984) are becoming very popular. So are the optimal techniques based on projection (Wilson 1970, Hyland and Bernsicin 1985) optimal methods for the reduction of time-delay models are also receiving desired attention (Halevi 1996, Taiwo 1999). These methods are also very powerful and of current research.

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