A Maximal Order Collocation Method for Direct Solution of Initial Value Problems of General Second Order Ordinary Differential Equations

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Abstract

In this paper, we propose a maximal method for direct solution of initial value problems of general second order differential equations. The method is based on collocation of the differential system at all grid points and interpolation of the approximate solution at $x = x_{n+j}$, $j = 0(1)3$. This approach yields an optimal order method, which is symmetric and therefore suitable for oscillatory initial value problems. Furthermore, a number of explicit schemes are proposed to cater for the values of y_{n+j}, $j = 1(1)4$, in the main method. The method is consistent and zero stable with a moderately large interval of absolute stability for non-stiff problems. Finally, the efficiency of the method for $k = 4$ is compared with Awoyemi [2001].

Key words: Collocation method; General Second Order Odes; Explicit method; Grid points: Interval of absolute stability.

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1. Introduction

The mathematical models of physical phenomena in many spheres of human endeavours often lead to one or system of nth order ordinary differential equations of the form

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}) \tag{1}$$

Xε [a, b]; f, y ε R and $y^{(s)}$, s = 0(1) n, denotes the nth derivative of the dependent variable y with respect to the independent variable *x*.

The commonest method for solving (1) is by reduction of the problem into a system of first order equations and then adopts any appropriate numerical method for first order equation to solve the system, (see Fatunla [1988], Lambert [1973]). However, the major drawback to such an approach is that the computer programme often developed to test the accuracy of such method is always found to be complicated and much of computer time is wasted. Apart from the main program the method may require other subprograms to handle the starting values for the evaluation of the functions arising from this system of equations.

In this paper, we propose a maximal method for solving the general second order initial value problem of the form

$$y'' = f(x, y, y'),\ y(x_0) = y_0,\ y(x_0) = \tau \tag{2}$$

Without reducing it to a system of first order equations. Some attention by eminent scholars has also been given to the problem. Henrici [1962] and Lambert [1973] discussed the theory of direct finite difference methods for solving (2). Hairer and Wanner [1976] developed Nystrom – type method in which order conditions for the determination of the parameters of the method were discussed. Gear [1971a], Hairer [1977], and Chawla and Sharma [1985] also developed independently explicit and implicit Runge – Kutta – Nystrom methods for numerical solution of (2). Many other contributors have delved into this subject but little seems to have been done in using collocation linear multistep methods (LMM) with continuous coefficients to solve directly problems of type (2).

Awoyemi ([1999], [2001]) proposed a class of continuous methods for (2). The results obtained showed an order 4 scheme for *k* = 2 and *k* = 3, and schemes of order 5 and order 6 for
k = 4. In this work, we develop a collocation algorithm that yields *a* maximal order scheme with continuous coefficients for solving the IVP (2).

2. Development of the Method

We propose a polynomial function

$$y_k(x) = \sum_{j=0}^{2k} a_j x^j \tag{3}$$

to be an approximate solution to (1.2), with the first and second derivatives as

$$y^1(x) = \sum_{j=1}^{2k} j a_j x^{j-1} \tag{4}$$

$$y^{11}(x) = \sum_{j=2}^{2k} j(j-1)a_j x^{j-2} \tag{5}$$

Collocating the differential system (5) at all the grid points and interpolating the approximate solution (3) at $x = x_{n+i,}$ $i = 0(1)\ 3$, yield

$$\sum_{j=2}^{2k} j(j-1)a_j x_{n+i}^{j-2} \quad = f_{n+i}; i = 0(1)4 \tag{6}$$

$$\sum_{j=0}^{2k} a_j x_{n+i}^{j} \quad = y_{n+i}; i = 0(1)3 \tag{7}$$

Where $f_{n+i} = f\,(x_{n+i}, y_{n+i,}\ y'_{n+i})$ and $y_{n+i} = y\,(x_{n+i})$, $i = 0(1)\ 4$.

Solving (6) and (7) for a_j's and substituting their values into (3), we, after some algebraic evaluation obtained a continuous scheme of the form

$$y_k(x) = \sum_{j=o}^{k-1} \alpha_j y_{n+j} + h^2 \sum_{j=o}^{k} \beta_j f_{n+j} \tag{8}$$

By letting

$$t \ = \ (x - x_{n+k-1})/h, \tag{9}$$

we obtain the continuous coefficient of (8) for $k = 4$ to be in power series as follows

$$\alpha_3(t) = \frac{1}{1302}\{1302 + 6131t - 11508t^3 - 6559t^4 + 1995t^5 + 2506t^6 + 694t^7 + 63t^8\}$$

$$\alpha_2(t) = -\frac{1}{1302}\{9720t - 20412t^3 - 12033t^4 + 3339t^5 + 4578t^6 + 1326t^7 + 126t^8\}$$

$$\alpha_1(t) = \frac{1}{1302}\{2349t - 6300t^3 - 4389t^4 + 693t^5 + 1638t^6 + 570t^7 + 63t^8\}$$

$$\alpha_o(t) = \frac{1}{126}\{120t - 252t^3 - 105t^4 + 63t^5 + 42t^6 + 6t^7\}$$

$$\beta_4(t) = \frac{1}{312480}\{1512t - 1092t^3 + 4291t^4 + 6552t^5 + 3374t^6 + 756t^7 + 63t^8\}$$

$$\beta_3(t) = -\frac{1}{156240}\{9216t - 78120t^2 - 165004t^3 - 70028t^4 + 31752t^5 + 31808t^6 + 8452t^7 + 756t^8\}$$

$$\beta_{2}(t) = -\frac{1}{156240}\{441288t - 1095444t^{3} - 647073t^{4} + 190260t^{5} + 244818t^{6} + 67752t^{7} + 6111t^{8}\}$$

$$\beta_{1}(t) = -\frac{1}{156240}\{192240t - 416724t^{3} - 191548t^{4} + 94248t^{5} + 75208t^{6} + 14652t^{7} + 756t^{8})$$

$$\beta_{o}(t) = -\frac{1}{312480}\{16344t - 33628t^{3} - 12971t^{4} + 9072t^{5} + 5306t^{6} + 484t^{7} - 63t^{8}\} \tag{10}$$

The first derivative of (10) is given by

$$\alpha_{3}^{1}(t) = \frac{1}{1302}\{6131 - 34524t^{2} - 26236t^{3} + 9975t^{4} + 15036t^{5} + 4858t^{6} + 504t^{7}\}$$

$$\alpha_{2}^{1}(t) = -\frac{1}{1302}\{9720 - 61236t^{2} - 48132t^{3} + 16695t^{4} + 27468t^{5} + 9282t^{6} + 1008t^{7}\}$$

$$\alpha_{1}^{1}(t) = \frac{1}{1302}\{2349 - 18900t^{2} - 17556t^{3} + 3465t^{4} + 9828t^{5} + 3990t^{6} + 504t^{7}\}$$

$$\alpha_{0}^{1}(t) = \frac{1}{126}\{120 - 756t^{2} - 420t^{3} + 315t^{4} + 252t^{5} + 42t^{6}\}$$

$$\beta_{4}^{1}(t) = \frac{1}{312480}\{1512 - 3276t^{2} + 17164t^{3} + 32760t^{4} + 20244t^{5} + 5292t^{6} + 504t^{7}\}$$

$$\beta_{3}^{1}(t) = -\frac{1}{156240}\{9216 - 156240t - 495012t^{2} - 280112t^{3} + 158760t^{4} + 190848t^{5} + 59164t^{6} + 6048t^{7}\}$$

$$\beta_{2}^{1}(t) = -\frac{1}{156240}\{441288 - 3286332t^{2} - 2588292t3 + 951300t^{4} + 1468908t^{5} + 474264t^{6} + 48888t^{7}\}$$

$$\beta_{1}^{1}(t) = -\frac{1}{156240}\{192240 - 1250172t^{2} - 766192t^{3} + 471240t^{4} + 451248t^{5} + 102564t^{6} + 6048t^{7}\}$$

$$\beta_o^1(t) = -\frac{1}{312480}\{16344 - 100884t^2 - 51884t^3 + 45360t^4 + 31836t^5 + 3388t^6 - 504t^7\} \tag{11}$$

When (10) is evaluated at $x = x_{n+4}$ or at $t = 1$, the result yields a symmetric scheme

$$\begin{aligned} &31y_{n+4} + 128y_{n+3} - 318y_{n+2} + 128y_{n+1} + 31y_n \\ &= \frac{h^2}{15}(23f_{n+4} + 688f_{n+3} + 2358f_{n+2} + 688f_{n+1} + 23f_n) \end{aligned} \tag{12}$$

With its first derivative given by

$$\begin{aligned} y'_{n+4} &= -\frac{12128}{651h}y'_{n+3} + \frac{15065}{434h}y'_{n+2} - \frac{2720}{217h}y'_{n+1} - \frac{149}{42h}y'_n + h\{\frac{265}{1116}f'_{n+4} + \frac{31708}{9715}f'_{n+3} \\ &+ \frac{311247}{19530}f'_{n+2} + \frac{49564}{9765}f'_{n+1} + \frac{7043}{39060}f'_n\} \end{aligned} \tag{13}$$

The scheme (12) is of order p = 8 and the principal error constant $c_{p+2} = -0.00013484$. The method is consistent and of maximal order, (see Lambert (1973, p.254)).

3. Region of Absolute Stability of the Method

In determining the region of absolute stability of the scheme (12), we adopt the boundary locus method Lambert [1973] to have

$$\bar{h}(\theta) = \frac{\rho\{\exp(i\theta)\}}{\sigma\{\exp(i\theta)\}} \tag{14}$$

Where ρ is the first characteristic polynomial and σ is the second characteristic polynomial.

This yields

$$\bar{h}(\theta) = \frac{15\{31\exp(4i\theta) + 128\exp(3i\theta) - 318\exp(2i\theta) + 128\exp(i\theta) + 31\}}{\{23\exp(4i\theta) + 688\exp(3i\theta) + 2358\exp(2i\theta) + 688\exp(i\theta) + 23\}} = x(\theta) + iy(\theta) \tag{15}$$

After some algebraic manipulation, we obtain

$$x(0) = \frac{15\{1426\cos 4\theta + 48544\cos 3\theta + 307696\cos 2\theta + 214624\cos\theta - 572290\}}{\{1058\cos 4\theta + 63296\cos 3\theta + 1163624\cos 2\theta + 6552512\cos\theta + 6507910\}}$$

and y(θ) = 0

The result is shown in the following tabulation

θ	0°	30°	60°	90°	120°	150°	180°
x(θ)	0	- 0.2742	-1.0966	-2.4654	-4.3443	-6.4157	-7.4708

This shows from the table that the region of absolute stability of the method (12) is $x(\theta) = $ (-7.47, 0) which is contained in the negative x-axis.

4. Development of Predictors

The following explicit schemes and their derivatives are developed to calculate y_{n+4}, y_{n+3} and y^1_{n+4}, y^1_{n+3} respectively

$$y_{n+4} = -16y_{n+3} + 34y_{n+2} - 16y_{n+1} - y_n + \frac{h^2}{3}\{8f_{n+3} + 44f_{n+2} + 8f_{n+1}\} \tag{16}$$

$$y'_{n+4} = -\frac{1124}{15}y_{n+3} + \frac{4399}{30}y_{n+2} - \frac{553}{10}y_{n+1} - \frac{127}{30}y_n + \frac{h^2}{45}\{400f_{n+3} + 2767f_{n+2} + 508f_{n+1}\} \tag{17}$$

$$y_{n+3} = 3y_{n+2} - 3y_{n+1} + y_n + h^2\{f_{n+2} - f_{n+1}\} \tag{18}$$

$$y^1_{n+2} = \frac{11}{2}y_{n+2} - 10y_{n+1} + \frac{9}{2}y_n + \frac{h^2}{6}\{11f_{n+2} - 23f_{n+1}\} \tag{19}$$

We also adopt Taylor Series expansion to express y_{n+j} and y^1_{n+j}, j = 1, 2 for use in (12), (13), (16), (17), (18) and (19), to have

$$\begin{aligned}
y_{n+j} &= y_n + (jh)y^1_n + \frac{(jh)^2}{2!}f_n + \frac{(jh)^3}{3!}\left\{\frac{\partial f_n}{\partial x_n} + y^1_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y^1_n}\right\} \\
&+ \frac{(jh)^4}{4!}\left\{\frac{\partial^2 f_n}{\partial x_n^2} + y^1_n\left(\frac{\partial y^1_n}{\partial y_n}\right)\left(\frac{\partial f_n}{\partial y_n}\right) + (y'_n)^2\frac{\partial^2 f_n}{\partial y_n^2} + y'\left(\frac{\partial f_n}{\partial y_n}\right)\left(\frac{\partial f_n}{\partial y'_n}\right) + y'_n f_n\frac{\partial^2 f_n}{\partial y_n\partial y'_n}\right\} \\
&+ \frac{(jh)^5}{5!}\left[\begin{array}{l}
\frac{\partial^3 f_n}{\partial x_n^3} + y^1_n\left(\frac{\partial y^1_n}{\partial y_n}\right)^2\left(\frac{\partial f_n}{\partial y_n}\right) + (y'_n)^2\left(\frac{\partial^2 y^1_n}{\partial y_n^2}\right)\left(\frac{\partial f_n}{\partial y_n}\right) + \left\{(y'_n)^2 + 2y'_n\right\}\left(\frac{\partial y^1_n}{\partial y_n}\right)\left(\frac{\partial^2 f_n}{\partial y_n^2}\right) \\
+ (y'_n)^3\left(\frac{\partial^3 f_n}{\partial y_n^3}\right) + y'_n\left(\frac{\partial y'_n}{\partial y_n}\right)\left(\frac{\partial f_n}{\partial y_n}\right)\left(\frac{\partial f_n}{\partial y'_n}\right) + (y'_n)^2\left(\frac{\partial^2 f_n}{\partial y_n^2}\right)\left(\frac{\partial f_n}{\partial y'_n}\right) \\
+ 2(y'_n)^2\left(\frac{\partial f_n}{\partial y_n}\right)\left(\frac{\partial^2 f_n}{\partial y_n\partial y'_n}\right) + y'_n f_n\left(\frac{\partial y'_n}{\partial y_n}\right)\left(\frac{\partial^2 f_n}{\partial y_n\partial y'_n}\right) + (y'_n)^2 f_n\left(\frac{\partial^3 f_n}{\partial y_n^2\partial y'_n}\right) + O(h^6)
\end{array}\right] + 0(h^6)
\end{aligned} \tag{20}$$

and

$$
\begin{aligned}
y'_{n+j} &= y'_n + (jh) f_n + \frac{(jh)^2}{2!}\left\{ \frac{\partial f_n}{\partial x_n} + y_n^1 \frac{\partial f_n}{\partial y_n} + f_n \frac{\partial f_n}{\partial y_n^1} \right\} \\
&+ \frac{(jh)^3}{3!}\left\{ \frac{\partial^2 f_n}{\partial x_n^2} + y_n^1 \left(\frac{\partial y_n^1}{\partial y_n} \right)\left(\frac{\partial f_n}{\partial y_n} \right) + (y'_n)^2 \frac{\partial^2 f_n}{\partial y_n^2} + y' \left(\frac{\partial f_n}{\partial y_n} \right)\left(\frac{\partial f_n}{\partial y'_n} \right) + y'_n f_n \frac{\partial^2 f_n}{\partial y_n \partial y'_n} \right\} \\
&+ \frac{(jh)^4}{4!}\left[\begin{aligned}
&\frac{\partial^3 f_n}{\partial x_n^3} + y_n^1 \left(\frac{\partial y_n^1}{\partial y_n} \right)^2 \left(\frac{\partial f_n}{\partial y_n} \right) + (y'_n)^2 \left(\frac{\partial^2 y_n^1}{\partial y_n^2} \right)\left(\frac{\partial f_n}{\partial y_n} \right) + \left\{ (y'_n)^2 + 2y'_n \right\} \left(\frac{\partial y_n^1}{\partial y_n} \right)\left(\frac{\partial^2 f_n}{\partial y_n^2} \right) \\
&+ (y'_n)^3 \left(\frac{\partial^3 f_n}{\partial y_n^3} \right) + y'_n \left(\frac{\partial y'_n}{\partial y_n} \right)\left(\frac{\partial f_n}{\partial y_n} \right)\left(\frac{\partial f_n}{\partial y'_n} \right) + (y'_n)^2 \left(\frac{\partial^2 f_n}{\partial y_n^2} \right)\left(\frac{\partial f_n}{\partial y'_n} \right) \\
&+ 2(y'_n)^2 \left(\frac{\partial f_n}{\partial y_n} \right)\left(\frac{\partial^2 f_n}{\partial y_n \partial y'_n} \right) + y'_n f_n \left(\frac{\partial y'_n}{\partial y_n} \right)\left(\frac{\partial^2 f_n}{\partial y_n \partial y'_n} \right) + (y'_n)^2 f_n \left(\frac{\partial^3 f_n}{\partial y_n^2 \partial y'_n} \right)
\end{aligned} \right] \\
&+ O(h^5) \qquad \qquad \text{.................}(4.6)
\end{aligned} \qquad (21)
$$

Where

$j = 1,\ 2$

$f_n = f(x_n, y_n,\ y'_n)$

$y_n = y(x_n)$

5. Numerical Examples

The following test examples are solved to illustrate the accuracy of our scheme (12) for the step size $h = \frac{1}{320}$. The results obtained are compared with that of Awoyemi [2001]. The errors arising from the computed results and theoretical solutions at some selected points are shown in Tables 1 and 2.

Example 1

$$2yy'' - (y')^2 + 4y^2 = 0, \qquad y\left(\pi/6\right) = \frac{1}{4}, \qquad y'\left(\pi/6\right) = \frac{\sqrt{3}}{2}.$$

Theoretical solution: $y = \sin^2 x$.

Example 2.

$$y'' = x(y')^2, \qquad y(0) = 1, \quad y'(0) = \frac{1}{2}.$$

Theoretical solution: $y = 1 + \frac{1}{2}\ln\left((2+x)/(2-x)\right)$.

TABLE 1: Errors arising from computed results to Example 1

x	Awoyemi [2001] for $k = 4$	New Method (12) for $k = 4$
1.1	0.4692146182D-06	0.416327943009165D-06
1.2	0.4080286853D-06	0.458667234304322D-06
1.3	0.2289737586D-06	0.409281963875685D-06
1.4	0.8128718132D-07	0.262954980123631D-06
1.5	0.5244721664D-06	0.455386863773555D-07
1.6	0.1089743773D-05	0.480547994174962D-06
1.7	0.1753725421D-05	0.103224610659147D-05
1.8	0.2481480677D-05	0.167849786836261D-05
1.9	0.3228415464D-05	0.238575061528579D-05
2.0	0.3943014561D-05	0.311083955728542D-05

TABLE 2: Errors arising from computed results to Example 2

x	Awoyemi [2001] for $k =4$	New Method (12) for $k = 4$
0.1	0.2607525307D-09	0.663913368725844D-13
0.2	0.1981670383D-08	0.200120364723944D-09
0.3	0.6507412165D-08	0.172007319498846D-08
0.4	0.1559238072D-07	0.589463944322688D-08
0.5	0.3150447658D-07	0.144346679054053D-07
0.6	0.5637457656D-07	0.418663523849006D-07
0.7	0.9616404606D-07	0.531096449041257D-07
0.8	0.1568680132D-06	0.911316806462281D-07
0.9	0.2486976918D-06	0.149241571545389D-06
1.0	0.3879838943D-06	0.237188791496479D-06

6. Conclusion

A maximal four-step continuous multistep method for general second order ordinary differential equations is developed. The method is of order $p = 8$ with principal error constant $C_{p+2} = -79/18900$. Numerical results of the method are compared with the sixth-order method developed by Awoyemi [2001].

An examination of Tables 1 and 2 clearly reveals that the new method is more accurate than that of Awoyemi [2001] for k = 4. The method (12) is also symmetric and has large interval of absolute stability, hence it will be useful for moderately stiff and non-stiff general second order ordinary differential equations.

7. References

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