Mathematical Modeling of the Exploitations of Biological Resources in Forestry and Fishery*

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Abstract

In this paper the fundamentals of modeling and simulation are discussed. Simulation experiments are designed for plant growth and population of Salmon fish. Problem on bionomic equilibrium of multi-species fish model of Kar-Chaudhri's([15]) including stability properties studied .The impulsive fish model of Pandit and Deo([13]) and Oyelami and Ale ([12]) type of fish and fish-hyacinth are considered including stability property of later investigated. The investigation is carried out using both quantitative and qualitative techniques with the simulation experiments designed using mathlab software.

Introduction

Different people understand model from different context. A Taylor will understand a model to be a piece of paper cut to represent a cloth to be sown. An architect sees a model to be a piece of cardboard paper cut and joined together to produce a miniature of a structure to be built.

Furthermore, a geographer scales down (draw-up) the plan of a city or town into what is referred to as a map. A map is a piece of paper drawn to represent infrastructures like rivers, rocks and important monuments in a place scaled down into the map. Hence, to a geographer a map is a model of a place.

The definition of a model has different definitions from different disciplines, but they all shear one thing in common. That is, the ability of being a miniature or 'scale down' of a physical object or a real life system into simple structure called –model ([3],[10]).

Here in our discussion we will look at a model from mathematical point of view, which is why we often use the word "Mathematical" to qualify a model in mathematical context.

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A mathematical model can be thought to be an abstraction of a real life system represented mathematically for the purpose of prediction and control of real life system. This simply mean replacing the real system with a simpler one, in mathematical sense, for purpose of understanding or mimicking the system. This is not a formal definition of a model, since its definition varies from one discipline to other. A model can also be said to be a mathematical structure with non-conflicting rules defined on it and used for purpose of understanding the real implication of the system([3],[10-11]).

Modeling is the act of developing models and studying them. Modeling activity has the advantage of being used to determine the effect of changes that the system will be susceptible to or amenable to due to some changes in the internal structure of the system. The internal structure in this context refers to the parameters or the variables used to quantity the model. We will discuss more on this when discussing steps taken to develop a mathematical model and the essential ingredients of a good model.

A model must have the following ingredients:

1. Ability to be realistic and simple as possible.
2. It must be a close approximation of the real system in a reasonable way and it must incorporate most important aspect of real system it seeks model.
3. Must not be complex, not impossible to understand and manipulated. Complex mathematical model may require complex mathematical or numerical methods to analyze it.
4. Must not contain unnecessary parameter(s) or too many assumptions that makes the model redundant and uninteresting. Assumptions and attestable hypotheses should not be contained in the model or made too simple by ignoring mechanism considered less important but are indeed necessary to understand the problem, the model seek solving.

Why Are Mathematical Models Needed?

As mentioned above, a good tailor cannot start cutting cloth without knowing the style or model needed to be sown or a builder start erecting a building without using the drawing plan of the structure to be built.

Mathematical models have been used to understand real life processes that may be dangerous to be undertaken directly, for example, nuclear reactors. We cannot directly experiment on them without building models to study the nuclear reactors process and know feasible conditions to operate the reactor plant. Space machines operate on nuclear fuel; they cannot undertake exploration without simulation on computer.

Mathematical models are ultimately used for prediction. Isaac Newton predicted that gravitational pull at the surface of moon, the earth's artificial sate lights is one sixth that of the Earth's (about 1.63 Newton). He arrived at this prediction through his universal law of gravitation. This is a kind of model postulated by Newton himself.

It is interesting to note that, in Newton's days there were no aircraft or space ship to go to the moon and conduct experiment to verify Newton's claim. Today many space explorations have been made to the moon and Newton's prediction proved to be true. Mathematical models are often used for discussing theoretical principles of real life processes and drawing conclusion concerning qualitative or phenomenal principles of dynamics underlying such processes.

Models have been used less successfully to provide accurate quantitative description and prediction that can serve as a testable hypothesis to rigorously confirmation with data ([5]).

As mention above, it is important; by identifying mechanism governing the real life system one can build simple models with few parameters. In case of ecological models, which are examples we will study, over abundance of parameters and state variables (relative to the amount of data available) are not statistically testable.

Models are idea tool for studying ecological problems, even though they are often corrupted with noise in the biological data quantifying them. Hence, some models have to be studied from probability context incorporating the idea of randomness and uncertainty. Models built in this direction are often called stochastic models otherwise deterministic models. Stochastic models are used to study stochastic processes.

Stochastic processes are series of events and state sequences representing the behaviour of a system. Stochastic models deal with models whose states are probabilistic in nature or are modeled with small variants while deterministic models as the name implies are models whose values can be determined concretely with time, e.g. population of given specie in a niche for example:

In building a good model one may have to build the two types of the models simultaneously. The deterministic to have exploration idea about the problem and to capture the problem, and then the stochastic model may be constructed to account for inevitable deviation from prediction or the deterministic model. Stochastic have been strongly advocated by ([5]) to explain the connection between a model and data.

Major steps needed to model a problem:

- Identification of the problem: The problem to be modeled need to be properly identified; we need to identify the variables, parameters and constants to be contained in the model.
- Formulation of the model we need to formulate the mathematical relationship that

exists between the variables.

- Investigate the model: Solve the formulated equation for concrete answer using the existing, techniques or appropriate method to validate the outcome, make empiric verification or to check whether the outcome is correct using the initial (baseline) data, if so, the model may be useful for prediction purpose, otherwise the model need to be modified or updated. This process is often being referred to as 'calibration' of model.
- Update of the model if necessary, the parameters, and constants in the model need by adjusted to get the right type of solution needed so that it can be used for prediction purposes.

The following diagram shows the fundamental steps needed to model a problem.

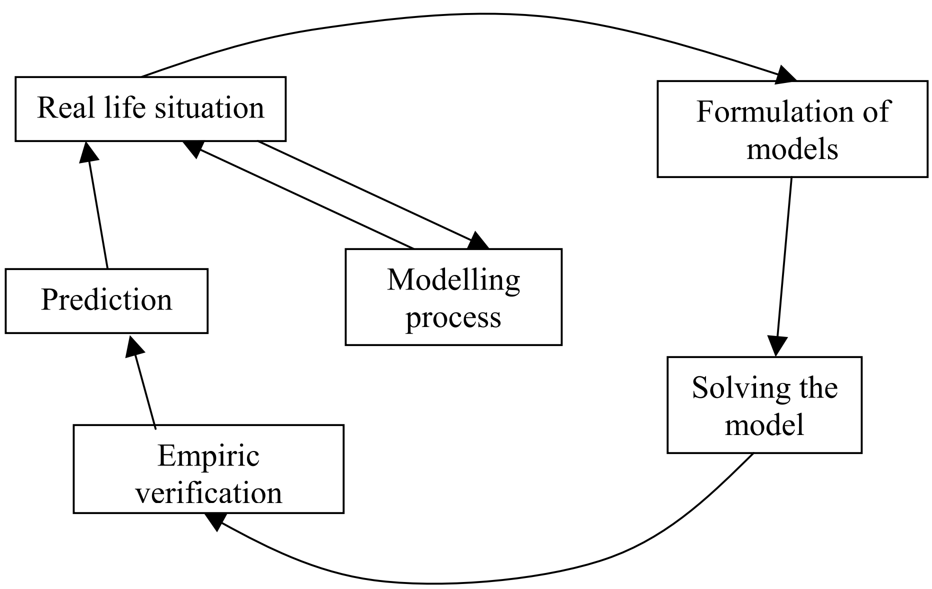


Fig. 1: Processes involved in modeling a problem.

The whole processes involved in modeling a problem is like a cycle, once a step is not properly carried out, the result will be faulty at the end of the day and one got to update the process to arrive at reasonable solution to the problem.

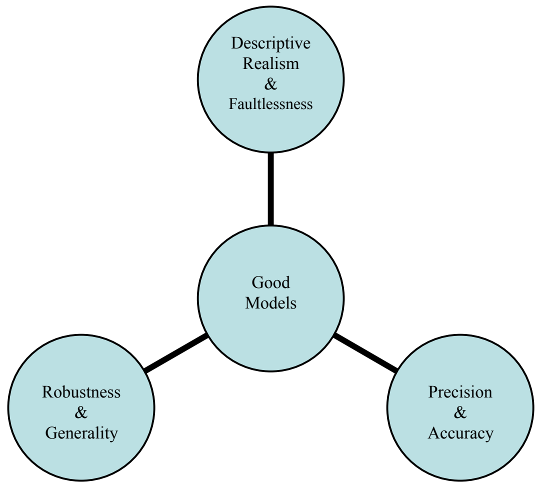
Systematization

The idea of systematization of data is getting data into coherent form and order. This may be merely a matter of careful arrangement and tabulation, or the data may be subjected to some mathematical manipulation using statistics and drawing graph etc.

Hypothesis

After experiment has been conducted and data systematized, the next ask is to show that new facts fit into the existing body of knowledge about the existing real life system. This can be established and conclusion at arrived at by testing the hypothesis.

Features of a good model



- Accuracy: That is its outcome must be correct or nearness to being correct.
- Precision: The prediction must be finite (that is have a definite number)

  Robustness: The model must be immune to errors in the input data.

  Generality: Can be adapted for general purposes and thus, not too specific in nature, hence, has global application in nature.
- Faultless: Conclusion drawn from it must useful and should provide the way for other models.

Simulation of Ecological Problem

The era of computer has championed the use of computational mathematics. The trend is how to apply computer in solving mathematical problems or the others. Computer can be said to the midwife of simulation. Simulation in simple layman language is the implementation of a model in a computer environment.

Simulation has been defined by C. West Churchman, amongst several definitions to be 'X simulates Y if and only if Y is taken to be an approximation to X', and Schubik defined it to be: 'simulation is therefore essentially a technique that involves setting up a model of the real life situation and performing experiments on the model'. In short simulation is a numerical technique for conducting experiments on a digital computer. It involves certain type of mathematical and logical modeling activities that describe the behaviour of a system over period of time ([10]).

The following flowchart (fig.3) illustrates the step taken to simulate problems:

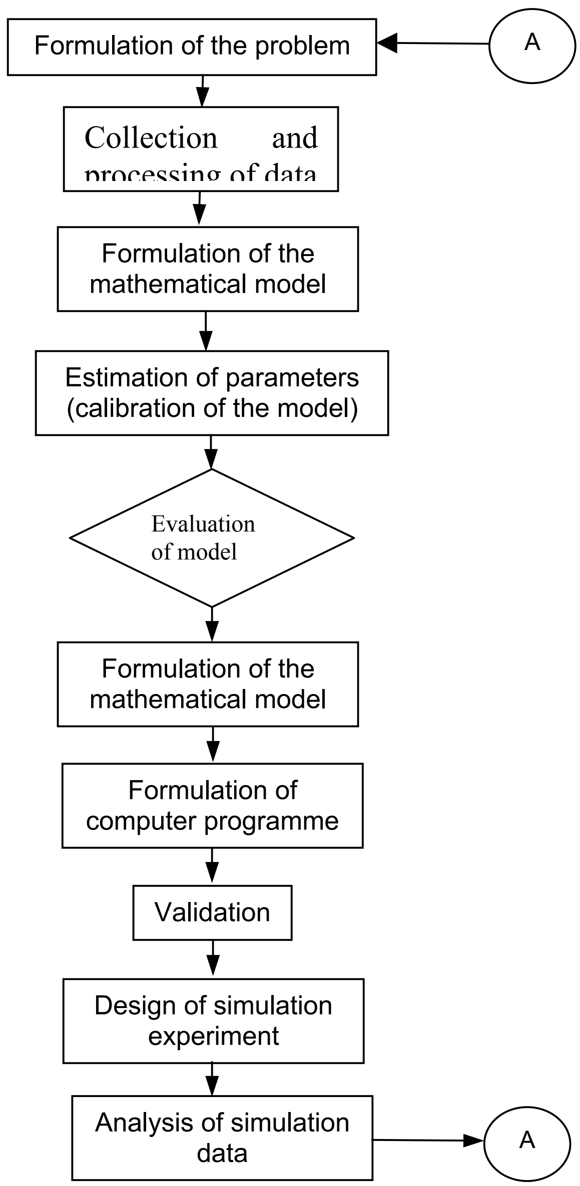


Fig. 3: Planning and Simulation Program

Simple mathematical modeling of a plant growth

Deforestation is one of the major causes of environmental problems. It is noted to be responsible for erosion of various forms.

A forestation has been advocated as a control of erosion. For this reason, sustainability of a forestation programmes should be aggressively pursued in Nigeria to checkmate erosion and other environmental menace.

A forestation, apart from being used for erosion control has the economic advantage of being used as cash crops and source of food supplies.

We will consider a simple modeling activity for the growth of a plant. We make use of Matlabs Version 5 as the simulation package to run the experiment, although, there are higher versions of the package available in the market. The experiment can still be run using the higher ones.

In all living organisms food is essential for growth and the two main sources of food for plant are carbon and nitrogen. There are also mineral salts in trace quantity needed for growth.

Carbon is supplied by the atmosphere and is absorbed through the leaves and combined with water in the presence of light (as catalyst) to produce sugar. Through, complex biochemical processes the sugar is converted to starch and other forms of compounds food products. This process is called photosynthesis.

Furthermore, nitrogen is supplied from the soil and is absorbed through the roots. The mechanisms by which plants absorbs, transport and combine these elements to produce growth will not be discuss here. The mathematics involved to model the plant is complex and involving. We will be concerned with simple mathematics surrounding the plant growth and how to use computer to study and monitor the growth.

The Model

This simple model is obtained if one does not distinguished between the roots and the shoots or between the carbon and nitrogen as food. Then one has a single variable which depends directly on how much food it can absorb.

We simply make reasonable assumption that the rate of absorption of carbon or nitrogen by the root or shoot is directly proportional to the volume of the plant.

If W is the weight of the plant and V is the volume.

We have the following variation relation

$$\frac{dW}{dt} \propto V \tag{1}$$

In rate equation form, this equal to

$$\frac{dW}{dt} = kV, \; k = constant$$

But V $= \frac{W}{\rho}$, ρ is the density of the plant, then eq. (1) leads to

$$W = W_o \exp\left(\frac{kt}{\rho}\right) \tag{2}$$

Where W_o is the weight when the experiment started, that is, when time is t = 0.
Eq. (2) suggests the growth pattern of the plant is exponential in nature, which, as time increases become infinite. No plant can grow indefinitely. Therefore, eq (2) cannot be realistically be used to model the plant growth. Since, prediction make with it will not be accurate. The solution of eq (1) becomes unrealistic as time increases, since as the plant becomes larger, it uses larger amount of maintenance and lower building energy. The model needs to be modified by assuming that k is no longer a constant but it is a variable that changes with respect to time. The simplest form of k would be linear i.e. k = a - 2bt, where *a* and *b* are constants and t is the time.

This will mean that equation (1) will become

$$\frac{dW}{dt} = (\alpha - 2bt)\frac{W}{\rho} \tag{3}$$

Where solution is valid for t < a/2b.
The equilibrium growth can be attained if $\frac{dw}{dt} = 0$
i.e. a - 2bt = o or t = a/2b.

For the period before equilibrium state is attained the solution to the model is more realistic and gives realistic approximation to growth.
The plant attains a maximum size at t = $\frac{a}{2b}$ at the equilibrium state and the maximum value being

$$W_o \exp (a^2 /4bp)$$

Let us ask the question: What happens if k = a -2bt + ct^2 or k = exp α t? where α, a, b and c is constants. Under this situation, the rate constant assumes nonlinear posture. We will investigate the two situations as simply case I and II respectively.

Case I if k = a - 2bt + ct^2

eq (1) becomes

$$\frac{dW}{dt} = (a - 2bt + ct^2)\frac{W}{\rho} \tag{4}$$

Solving eq (4), we get

$$W = W_o \exp\left(\frac{1}{\rho}\int_o^t (a - 2bs + cs^2)ds\right)$$

$$= W_o \exp\left(\frac{1}{\rho}(at - bt^2 + \frac{c}{3}t^2)\right) \tag{5}$$

Case II

When

k = exp αt

We get

$$W = W_o \exp\left(\frac{1}{\rho}\int_o^t \exp \alpha s ds\right) = W_o \exp\left(\frac{1}{\alpha\rho}(\exp \alpha t - 1)\right)$$

Assignment

Determine the maximum values for W in eq (4) and eq (5) and the condition for realistic growth for the equation.

Age of a plant with circular annual ring growth the age of a plant, which forms annual ring can be computed from

$$n = \frac{W}{\pi r^2 h} \tag{6}$$

n = is the number of annual rings forms by the plant.

r = radius of the circular rings formed by the plant, assuming that radius is constant.

For realistic growth of a plant we may assume that

$$W = W(t,r,h) = ka(t,r,h)h(t) \tag{7}$$

that is, the weight depends on the annual radius, the height of the plant at time. The circular areas also depend on t, r, and during evolution period, and height is also time dependant.
Therefore

$$\frac{dW}{dt} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial r}\frac{dr}{dt} + \frac{\partial W}{\partial h}\frac{dh}{dt}$$

$$= \quad \frac{kh\partial a}{\partial t} + \left(\frac{ka\partial a}{\partial h} + ka\right)\frac{\partial r}{\partial t} + \left(2ka + kh\frac{\partial a}{\partial h}\right)\frac{\partial h}{\partial t}$$

it implies that

$$\frac{khda}{dt} + \frac{kadh}{dt} = kh\frac{\partial d}{\partial t} + \left(kh\frac{\partial a}{\partial h} + ka\right)\frac{\partial r}{\partial t} + \left(2ka + kh\frac{\partial a}{\partial h}\right)\frac{dh}{dt}$$

This can be expressed in a partial differential equation form as

$$F\left(t, a, h, r, \frac{\partial a}{\partial t}, \frac{\partial a}{\partial h}, \frac{\partial r}{\partial t}\right) = 0 \tag{8}$$

Where

$$F\left(t, a, h, r, \frac{\partial a}{\partial t}, \frac{\partial a}{\partial h}, \frac{\partial r}{\partial t}\right) = \frac{hda}{dt} - (a+h)\frac{dh}{dt} - h\frac{\partial a}{\partial t} - \left(h\frac{\partial a}{\partial h} - a\right)\frac{\partial r}{\partial t}. \tag{9}$$

The equation (1) can be verified experimentally only in the laboratory where periodic weighting of the plant can be made. For a field experiment, we assume that $W \propto ah$, $W =$ k_1ah, k, = rate constant
Therefore,

$$\frac{dW}{dt} = k\frac{dh}{dt}\alpha + kh\frac{da}{dt} = k_1\frac{ha}{\rho}$$

$$\frac{kdh}{hdt} + \frac{da}{adt} = \frac{k_1}{\rho}$$

$$\frac{kd\log(h)}{\rho dt} + kd\frac{(\log a)}{dt} = \frac{k_1}{\rho}$$

Solving the equation above, we arrive at

$$ah = a_o h_o \exp\left(\frac{k_0}{k\rho}(t - t_0)\right)$$

or

$$W = W_0 \exp\left(\frac{k_1}{kp}(t - t_0)\right) \tag{6}$$

Designing a Simulation Experiment for the plant growth

It is not very easy to determine W experimentally in the field, except in a laboratory where the experiment can be conducted using pots. Weighing of the pots can be done from time to time to determine the increment in weigh of the plants.

In the field experiment we rather use the height of the plant, which can be measured from time to time. We assume that $W \propto ah$, a = the cross section area of the plant, h = height of the plant.

We will soon discover that the value arrived at in the field experiment and the laboratory one are the same mathematically under certain conditions.

Consider a hypothetical situation in which a particular plant is being experimented upon. The growth being monitored, at time t, a, h and W are also recorded as follows:

Table 1: Hypothetic experimental result for plant growth

Table 1: Hypothetic experimental result for plant growth

t	0	1	2	3	4	5	6
A	0.1	1.7	3.3	4.00	4.43	5.13	5.6
H	0.1	3.3	5.0	83.3	8.83	9.50	10.1
W	0.01	5.6	16.5	33.3	39.13	48.74	8.74

Tah and *W* are measured in standard units.

To run the experiment, we wish to determine the rate growth for a, h and W using linear, quadratic or cubic rates of growth. We will use the Mathlab toolbox.

Box A.
p = polyfit (x, y, n) to find the coefficient of a polynomial p(x) degree is to fit the data p(x(i)) to y(i_).

[P, S] = polyfit (x, y, n) returns polynomial coefficient P, and Matrix S for W with polynomial to produce error estimates on prediction.

We will use polytool (x, y, n) initial fits a polynomial of order n polytool (x, y, alpha) plots 100 (1 - alpha) %. Confidence interval on a predicted value.
Polytool fits by least squares assume the regression model of the form:

$$y_i = \beta + \beta_1 x_1 + \beta_2 x_i^2 + \ldots + \beta x_n^n + \xi_n$$
$$\xi_i \approx N(0, \delta^2)$$

For all i

$$\text{Cov } (\xi_i \xi_j) = 0 \text{ for all i, j.}$$

N(o, δ^2) is normal distribution ith 0 mean and δ standard deviation and Cov$(\xi_i \xi_j)$ are the covariant vector. The toolbox A can be used to plot the graph of a function.

Box B

We use this Matlab toolbox, dsolve to solve the differential equations for the plant growth. For example

W=dsolve ('DW = k*W') returns the solution of the differential equation

$$\frac{dW}{dt} = kW \text{ at prompt} >>$$

Experiment 1

Using matlab window prompt type

```
            >> t = [0  1  2  3  4  5  6], a= [0.1  1.7  3.3 ...  5.6]; returns
      ans =
      t      =      0  1  2  3  4  5  6
      a      =      0.1  1.7   3.5  4.0 ...  5.6
>>    W      =      [0.01  5.6  10.5  33.3  39.13 ... 56.56] returns
      ans
      W      =      0.01  5.6  16.5  33.3  39.13  48.74  56.56
>>    % linear polynomial for a, h and W
      returns
      >>
```

```
>>    polyfit (t, a) return
>>    0.8740        0.8418
>>    polyfit (t, h) return
>>    1.6511        1.4982
>>    polyfit (t, w) return
>>    2.5257     2.3400
```

From above experiment a, h, and w can be represented linearly as

a = 0.8746t + 0.8418

h = 1.6511t + 1.4982

w = 2.5257t + 2.3400

The experiment can be repeated as above for the quadratic and cubic representation for a, h and w using

Polyfit (t, a, 2), polyfit (t, h, 2) and polyfit (t, w, 2)

For quadratic form

polyfit (t, a, 3) polyfit (t, h, 3) polyfit (t, w, 3)

See table 2 for the results.

To obtain graphic representation of the growth profile of the plant, we can plot the graph W against t.

Table 2: Quadratic and cubic rate growth of the plant

	Quadratic	Cubic
a	$-01273t^2 + 1.6382t + 0.2055$	$0.02610t^3 - 0.3623t^2 + 2.1604t + 0.0488$
h	$-0.2835t^2 + 3.3516t + 0.0210$	$-0.028t^3 - 0.2760t^2 + 3.3351t + 0.0880$
W	$-0.4107t^2 + 4.9900t + 0.2804$	$0.0253t^3 - 0.6882t^2 + 5.4956t + 0.1308$

For graphic display of W are use polytool (t, w, 3) and polytool (a, w, 4) to study the growth of W with respect to t, h and a respectively. See fig. 4 & fig.5, for the plot of the graphs.

Experiment 2

We use the toolbox in Box B to obtain the situation analysis of the model.

From experiment 1, the rates for the plant growth was obtain for W for linear, quadratic and cubic rates were found to be

k_l = 2.527t + 2.3400

k_q = $-0.4107t^3 + 4.9900t + 0.2804$

k_c = $0.0253t^3 - 0.06382t^2 + 5.4956t + 0.01348$

Thus, application of Matlab on the window prompts >>, we type in

```
>> % linear growth rate
>> p = 1.58, dsolve ('DW = (12.5257 * t + 2.3400) * W/ρ');
returns
ρ = 1.58
ans
C1* exp (3/2000*t*(8419 * t + 15600/ρ))
>> % quadratic growth rate of W
>> W = dsolve ('DW = (-0.4107*t ^2 + 4.9900*t + 0.2004)* W/ρ');
ans W =
C1*exp (-1/10000*t*(1369*t^2 - 24950*t - 2804)/ρ)
>> % cubic growth rate of W
>> W = dsolve ('DW = (0.0253*t^3 - 0.6882*t^2 + 5.4958*t + 0.01348)*W/ρ');
Ans W =
C1*exp (1/200000*t* (1265*t^3 - 4.580*t^2 + 549560*t + 2695)/ρ)
>> % exponential growth rate
>> Sym alpha, dsolve ('DW = (exp (alpha*t)*W/ρ');
returns
alpha
ans
C1*exp (1/ρ/ alpha*exp (alpha*t))
```

Through the experiment C_1 is constant of integration for the (Ode) which can be obtain from initial condition and for t = 0.Then $C_1 = W_o$. % is symbol for non-executive statement or remark statement.

From the experiment results, the weight of the plant tends to approach unbelievable large value as $t \to \infty$ for higher order rate relation for k.

Mathematical Models in Fishery

Mathematical models for exploitation of biological resources like in fishery and forestry are gaining applications these days. Mathematical modeling has found application in fishery. Oyelami and Ale [13], considered the Fish-Hyacinth model using the B-transform to obtain results on asymptotic growth value of the dual population under impulsive regime. That is, times for which the process are characterized by shocks, jumps and rapid changes that are have been proved to characterize most impulsive processes ([1-2], [4], [9]&[14]).

Cark discussed the problem of combined or non-selective or harvesting of two ecologically independent populations obeying the logistic laws of growth ([15]). Leslie models will be used to study the population of salmon fish.

Leslie Model

Leslie model describes the population of given specie, considering only the female population of the reproductive age group. Leslie in people developed this model in 1974 and used it to forecast population of people stratified into age groups. This model can be adapted for ecological purposes. It uses matrix algebra and it is a popular method because it allows some important ecological processes to be captured; survivorship and fertility.

The usefulness of Leslie model inculcating intrinsic rate of increase for predicting of proportion of each age group in a stables age distribution and prediction of the effects of change in the population dynamic brought about by change in survivorship and fertility.

The model can be written as

$$x^{(k)} = Ax^{(k-1)}$$

Where

$$A = \begin{pmatrix} F_1 & F_2 & \dots & F_n \\ P_1 & 0 & \dots & 0 \\ 0 & P_2 & \dots & 0 \\ & & P_{n-1} & 0 \end{pmatrix} \qquad (12)$$

The Leslies matrix
F_l, the average reproductive rate of a female in the *i*th age groups
P_i, is the survival rate of female in the *i*th age class.
The initial age distribution vector on the age distribution vector at time $t = t_o$
The age distribution vector at time $t = t_0$.

$$x(0) = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ . \\ . \\ . \\ x_n^{(0)} \end{pmatrix}$$

The solution to Leslie iterative equation is $x^{(n)} = A^n x(0)$ provided that $\max|\lambda(A)| < 1$ that is the maximum value of the absolute value of the eigenvalue of matrix A is less than unity. Let us apply this model to population of Salmon fish. Since the population of fish cannot be negative, we have $F_i \geq 0$ and Salmon $F_1 = 0$, $F_2 = 0$ because the Salmon only produce offspring in the last year of life. Thus, only F_3 has a positive value. Also, $0 < P_i \leq 1$, for i = 1, 2 that is, some Salmon must survive to the next age class, but must die after spawning.

$$x(0) = \begin{pmatrix} 1000 \\ 1000 \\ 1000 \end{pmatrix}$$

Survival rate in the first age group is 0.5% second age group is 10% and each female in the this age group produces 2000 female offspring. Then $P_2 = 0.005$, $P_1 = 0.10$ and $F_3 = 2000$.

Therefore $$A = \begin{pmatrix} 0 & 0 & 2000 \\ .005 & 0 & 0 \\ 0 & .01 & 0 \end{pmatrix}$$

Using Matlab, the problem can be simulated s follows:

```
>>      xo = [1000; 1000];
>>      A = [0  0  2000; 0.005  0  0 ;0. 10  0];
>>      X = zero (3, 24)
>>      X (:, 1) = xo
>>      For k = 2:24, X (: k) = A * X (: k - 1); end and the content of X can be display as
>>      X =
        Column 1 through 6
             1000   2e + 006     2e + 005     1000   2e + 006     2e + 005
             1000       5        10000        1000   5            10000
             1000     100        0.5          1000   1000         0.5
        Column 7 through 12
             1000   2e + 006     2e + 005     1000   2e + 006     2e + 005
             1000       5        1000         1000   5            1000
             1000     100        0.5          1000   100          0.5
        Column 12 through 19
        .
        .
        .
        Column 19 through 24
             1000   2e + 006     2e + 005     1000   2e + 006     2e + 005
             1000       5        1000         1000   5            1000
             1000     100        0.5          1000   100          0.5
```

We can plot the row of X versus the index but plot (X) plots the column of X versus the index. The solution: plot the transpose of X.

```
>>      Plot (X/)
```

Fig. 1: The Salmon population over time for semi logarithm plot use
>> semilogy (X^1)

Fig. 2: Semilog plot of Salmon population over time.
For legend type

```
>>      legend ('...' '...')
```

Mathematical modeling for exploitation of biological of the predator

Chaudhuri [15] studied the problem of combined harvesting of two competing species. Multi-species harvesting model and studies by Mesterten – Gibbons(see[15])

We consider Kar-Chaudhuri model

$$\frac{dx}{dt} = rx\,(1 - \frac{x}{K}) - \frac{m\,x\,y}{a + x} \qquad (13)$$

$$\frac{dy}{dt} = sy\,(1 - \frac{y}{L}) + \frac{m\,x\,y}{a + x} \qquad (14)$$

Where

x = x(t) = size of the prey population at time t.
y = y(t) = size of the predictor population at time t
K = environmental carrying capacity.

L = environmental carrying capacity of the predator.
M = maximal relative increasing of predator
A = Michael - Menten constant
α = Conversion factor (we assume $\alpha < 1$) since the whole biomass of the prey is not transformed to the predator.
R = intrinsic growth rate of the prey
S = intrinsic growth rate of the predator

If $$f((x, y) = \frac{mxy}{a + x}$$

Hollings has shown that, as t → ∞, f → mx that is the tropic, nor the function response of the predator to the density of prey.
Analytic solution to Kar-Chaudhari's model is intractable generally. As test let us use the matlab by attempting to solve the problem using the dsolver.
At dos prompts type in

```
>>      dsolve ('Dx = r*x*(1 - x/K) - m*x*y)/(a + x)',
                Dy = s*y* (1 - y/L) + (n*x*y)/(a + x)');
```

returns

Warning statement after long run time. The problem can however be solved using numerical solvers in the Matlab using the Runge-Kutta family and other solvers that are designed to handle stiff systems.

To determine equilibrium state (biological equilibrium)

$$\text{Set } \frac{dx}{dt} = 0, \frac{dy}{dt} = 0$$

This involves solving

$$\begin{aligned} rx(1-\frac{x}{K}) - \frac{mxy}{a+x} &= 0 \\ sy(1-\frac{y}{L}) + \frac{mxy}{a+x} &= 0 \end{aligned} \tag{15}$$

Using Matlab for particular values of r, s, m, a ,K and L the equilibrium state x* = x, y* = y in the above equation can be obtain from

$$>> solve('r*x*(1-\frac{x}{K})-(m*x*y)\Big/(a+x)=0','s*y*(1-\frac{y}{L})-{m*x*y}\Big/{(a+x)}=0');$$

Since, the analytic solution to model including the biological equilibrium cannot be obtained in closed form. We use qualitative techniques, that is, study the behaviour of the system rather than the solution to the model directly.

To investigate the stability of the model, we use the Lyapunov stability technique.

Now define $q_1 Ex = \frac{mxy}{a+x}, q_2 Ey = \frac{mxy}{a+x}$

q_i ,i=1,2 are the catchability coefficients of the two species.
Let

$$V(x,y) = \begin{bmatrix} r - \frac{2r}{K}x - \frac{amy}{(a+x\ )^2} - q_1 E & \frac{-mx}{(a+x)} \\ \frac{am\alpha y}{(a+x)^2} & s - \frac{2s}{L}y + \frac{amx}{(a+x)} - q_2 E \end{bmatrix} \tag{16}$$

The biological equilibrium points $(0,0),(\bar{x},0),(0,\bar{y}),(x^*,y^*)$
where

$$V(x^*, y^*) = \begin{bmatrix} -\frac{r}{K}x^* + \frac{mx^*y^*}{(a+x^*)^2} & \frac{-mx^*}{(a+x^*)} \\ \frac{amy^*}{(a+x^*)^2} & \frac{-sy^*}{L} \end{bmatrix} \quad (17)$$

It was proved (see [15]) that necessary and sufficient conditions for stable node is that

$$E > (\frac{r}{q_1}, \frac{s}{q_2}) \quad (18)$$

The economic equilibrium state of the model is said to be achieved when $\dot{x} = 0, \dot{y} = 0$ and when TR (the total revenue obtained by selling the harvested biomass) equal TC (Total Cost for the effort devoted to harvesting). Then revenue generated from selling the fish being

$$R = p_1q_1xE + p_2q_2yE - CE \quad (19)$$

Where

p_i are constant selling prices of x and y fishes.
C is the constant, which is the fishing effort costs.
For biological equilibrium

Set $\dot{x} = 0, \dot{y} = 0$

Thus occur when the following condition is satisfied

$$\frac{r}{Kq_1}x^2 - \frac{s}{Lq_2}xy - (\frac{r}{q_1}\frac{s}{q_2} + \frac{ma}{q_2})x + (\frac{m}{q_1} - \frac{as}{Lq_2})y - (\frac{ar}{q_1} - \frac{as}{q_2}) = 0 \quad (20)$$

For economic equilibrium

$$R = TR - TC = (P_1q_1x + P_2q_2y - C)E = 0 \quad (21)$$

It implies that

$$P_1q_1x + P_2q_2y - C = 0 \quad (22)$$

Equations (20) & (21) gives

$$A_1 x^2 + B_1 x + C = 0$$

Where

$$A_1 = \frac{r}{Kq_1} + \frac{s}{Lq_2}\frac{p_1 q_1}{p_2 q_2} > 0 \tag{23}$$

$$B_1 = \frac{s}{q_2} - \frac{r}{q_1} + \frac{ar}{Kq_1} - \frac{ma}{q_2} - \frac{Cs}{Lq_2}\frac{1}{p_2 q_2} - \left(\frac{n}{q_1} - \frac{as}{Lq_2}\right)\frac{p_1 q_1}{p_2 q_2} \tag{24}$$

$$C_1 = \frac{C}{p_2 q_2}\left(\frac{m}{q_1} - \frac{as}{Lq_2}\right) + \left(\frac{as}{q_2} - \frac{ar}{q_1}\right) \tag{25}$$

We can show that bionomic equilibrium exists if $B_1^2 - 4\,A_1 C_1 \geq 0$ and no bionomic equilibrium exists if

$$B_1^2 - A_1 C_1 = o.$$

Mathematical Models Involving Impulses

Impulsive differential equations are systems that are characterized by small perturbation inform of jumps ([1], [4],[9]) such systems are noted to be more general than system found in ordinary differential equations (Odes).

The underlying assumption of continuity and integrability of most systems, which form the crux of most hypotheses in Odes, we in general not true. Certain systems are susceptible to discontinuous changes in form impulses that are not integrable in the ordinary sense. For example, population is not continuous it is often susceptible to rapid changes which are continuous ([12],[14]).

Many processes in real life are impulsive whether biological, physical or economic in nature. For example Pandit and Deo ([13]) gave the following measure differential equation (m.d.e.): Consider a fish-breeding pond stocked with fishes. If some varieties of fish breed is released into a pond and allowed to grow. After some fixed time intervals, t_1, t_2 ... partially grown fish are removed from the pond. The growth of fish population is impulsive. The impulses are given at times t_1, t_2 ... This problem can be solve satisfactory using this equation

$$Dx(t) = \alpha x(t)\,Du, x(t_o) = x_o$$

Where α is some growth constants and D is the distributional derivative, u is a right continuous function of bounded variation.

Suppose that u (t) is taken as (23)

$$u(t) = t + \sum_{k=1}^{\infty} a_k\ H_k(t)$$

Where

$$H_k(t) = 1 \text{ if } t < t_k$$
$$- 0 \text{ if } t \geq t_k$$

And a_k is real numbers.

The model is a measure differential equations (m.d.e.) analogue of Malthusians equation if $d_1 = 0$, the problem reduce to a typical Malthusian equation in ordinary differential equations. This in clear indication that impulsive system generalizes some results in ordinary differential equations.

For further insight in generality of impulsive systems to Odes (see [4], [9], and [14]).

Fish-Hyacinth Model

Consider the differential - difference equation governing the modified fish-hyacinth model (see [12])

(IDDE)
$$\dot{x}_t = x_o + B_t - D_t - I_t - \alpha y_t, t \neq t_k(x_t, y_t)$$
$$\dot{y}_t = \frac{\mu y_t}{K}(K - y_t^p) + \beta x_t, t \neq t_{ik}(x_t, y_t)$$
$$\Delta x|_{t = T_k(x_t, y_t)} = x_{t_{t+o}} - x_{t_{k-o}} = -I_1(x_{t_k}), k = 1, 2, \ldots$$
$$\Delta y|_{t = T_i(x_{t_k} y_{t_k})} = y_{t_k+0} - y_{t_k-0} = I_2(y(t_k))$$

For strictly increasing sequence of times $t_k \varepsilon J_+ = [0, +\infty)$ such that $0 < t_0 < t_1 < t_2 < \ldots < t_k$, $\lim\limits_{k \to \infty} t_k = \infty$ where I_k for i = 1, 2 are the quantity of the biomass added (if $I_i < 0$) for i = 1, 2 and taken away from the environment if ($I_1 > 0$) at certain non-fixed moments t_i, i = 1, 2.

x(t) is the quantity in square units of the hyacinth present at time t and resume to be impulsive because of abrupt change of state.

B_t is the quantity of the hyacinth that germinates per unit square at time t with lag h.

D_t is the quantity of the weeds removed at the time t mechanically or by other means not and being fed upon by the fish.

A_t is the quantity being fed upon at time *t* by the fish.
y_t is the number of the fishes at time to present in the biome at time *t*.
K is the saturation point for the fish population, which the environment can support, and μ is the difference between the births - death rates of the fishes.

The factors
$-\alpha y_t$ and βx_t accounts for the inhibition of the fish growth and accelerated growth of the hyacinth respectively.

The problem is that of investigating the asymptotic stability of the null solution of the model.
Now, define

$$V(t, x_t, y_t) = \sum_{k=o}^{N}\sum_{l=o}^{N} a_k a_l x_t^2 + 2\sum_{k=o}^{N} a_k x_t y_t + y_t^2$$

That is, define

$$V(t, x_t y_t) = Ax_t + 2Bx_t y_t + Cy_t^2 + Dx_t + Ey_t + F$$

Where

$$A = \sum_{k=o}^{N}\sum_{l=o}^{N} a_k a_l, B = \sum a_k, C = 1, D = E = F = 0$$

Differentiating V (t, x_t, y_t) with respect to time along the solution path.

$$\dot{V} = \sum_{k=0}^{N}\sum_{l=0}^{N} a_k a_l x_t \dot{x}_t + 2\sum a_l \dot{x}_t y_t + 2\sum a_k x_t \dot{y}_t + 2y_t \dot{y}_t .$$

Therefore

$$\dot{V} = A^{|} x_l^2 + 2B^{|} x_t y_t + C^{|} y_t^2 + + E^{|} y_t + F$$

Where

$$A^{|} = 0$$
$$B^{|} = (\sum\sum a_k a_l \frac{\mu}{K} + \frac{2\mu N}{K}\sum a_k)(K - y_t^p)$$
$$C^{|} = \frac{2\mu}{K} N \sum a_k (K - y_t^p)$$
$$D^{|} = \sum\sum a_k a_l N(\dot{B}_t - \dot{I}_t - \dot{D}_t) \rightarrow S_t = constant$$
$$E^{|} = \frac{-2\mu N}{K}\sum a_k (K - y_t^p).$$

If then V is negative definite

$$(\sum\sum a_k a_l \frac{\mu}{K} + 2N\frac{\mu}{K}\sum a_k)(K - y_t^p) \geq -2\sum a_k$$

$$\frac{2N\mu}{k}\sum a_k(K - y_t^p) \geq -1$$

$\sum\sum a_k a_l N(\dot{B}_t - \dot{I}_t - \dot{D}_t) = 0$ it implies that $\dot{S}(t) = 0$ that is $S(t) = constant$

$$E = \frac{-2\mu N}{k}\sum a_k(K - y_t^p) = 0$$

$$F = 2N\sum a_k(\dot{B}_t - \dot{I}_t - \dot{D}_t) = 0$$

From equations (3 - 5) we have

$$B_t - I_t - D_t = constant$$

$K - y_t^p = 0$ *which* Implies

$$y_t = K^{\frac{1}{p}}$$

Equation (6 and 7) simply states that as regeneration function $B_t - I_t - D_t$ approach infinity it becomes constant and y tends to values $\boldsymbol{K^{\frac{1}{p}}}$. This condition gives sufficient condition for negative definiteness of V.

$$a^* = \max_k [1, \sum a_k], a_* = \min_k [1, \sum a_k]$$

And let

$$\text{Lim} \sum_{k\to\infty} a_k = a > 0$$

$$V(t, x_t, y_t) = (\sum a_k x_t + y_t)^2 \leq a^{*2}(x_t + y_t)^2$$

$$\leq |a^*|^2(|x_t + y_t|^2)$$

$$\leq |a^*|^2(|x_t|^2 + |y_t|^2)$$

$$= |a|^2(|z_t|)^2 = b(|z_t|^2)$$

Similarly

$$V(t, x_t, y_t) \geq -a_*^2(|x_t|^2 + |y_t|^2) = a(|z_t|)$$

Applying equation (1 - 7) to $\dot{V}$ (t, x_t, y_t) we find that

$$\dot{V}(t, x_t, y_t) \leq -V(t, x_t, y_t) \leq -C(V(t, x_t, y_t)).$$

The Fish-Hyacinth model will have stable biological equilibrium when

$$\dot{x}_t = 0, \dot{y}_t = 0, \Delta x_t = 0, \Delta y_t = 0.$$

To determine region of stable attractors
Let

$$\sum_{l=o}^{N} a_l x_t^2 = C(V_k(x_t))$$

$$C(V_s(y_t)) = y_t^2$$

$$\begin{aligned} C(V(t,x_t,y_t)) &= \sum_{k=o}^{N}\sum_{l=1}^{N} a_k a_l x_t^2 + y_t^2 \\ &= C(V_s(y_t)) + \Sigma a_k C(V_k(x_s)) \end{aligned}$$

Clearly

$$\dot{V}(t,x_t,y_t) \le - C(V(t,x_t,y_t))$$

Let define set of attractors by $A^c_{k\mathrm{B}}$ where

$$A^c_{k\mathrm{B}} = \{ (x_t, y_t) : \alpha x_t + \beta y_t + C \le 0, \alpha, \beta, C \,\varepsilon\, R \}$$

If $\quad \Delta x_t \le \beta, \Delta y_t \le - \beta_t$

Then

$$\begin{aligned} V(t_k + 0, x_t + \Delta x_t, y_t + \Delta y_t) &\le V(t,x_t,y_t) \\ &- 2(A - \beta\beta_1)x_t + 2(\beta\beta_2 - \beta_2)y_t \\ &+ 2A\beta_1^2 + \beta\beta_1\beta_2 + \beta_2^2 - \beta_1 + E\beta_2 \\ &\le \varphi(V(t,x_t,y_t)) \; for \; (x_t, y_t) \,\varepsilon\, A^c_{\alpha\beta}, \varphi\varepsilon\mathrm{K} \end{aligned}$$

Where

$\alpha \quad = \quad -2 (A - \beta\beta_1)$

$\beta \quad = \quad 2 (\beta\beta_1 - \beta 2)$

$C \quad = \quad 2A\beta_1^2 + 2\beta\beta_1\beta_2 + \beta_2^2 - \beta_1 - E\beta_2$

$\mathrm{K} = \{ \text{all monotone increasing continuous functions } \varphi : [0,\infty) \to [0,\infty) \}$

Then the solution of Fish-hyacinth is stable in $A^C_{\alpha\beta}$ for value of α, β and C given above. The implication of this is that the populations of the fish and that of hyacinth can be controlled in the long run.

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References

[1] Ale, S. O. and Oyelami, B. O. (2000)
Impulsive systems and their applications. Int. J. Math. Educ. Sci Tech., Vol. 31, No. 4, 539 – 544.

[2] Ale, S. O., and Oyelami, B. O. (2000)
B-Transform method and its applications in obtaining solutions of some impulsive models. Int. J. Math. Educ. Sci Tech., Vol 31, No. 4, 525 – 538.

[3] Andrew, J. G. and McCone, A. A. (1976).
Mathematical modeling. British worth and Company.

[4] Bainov, D. D. and Simeonov, P. (1989)
Impulsive Differential Equations: Asymptotic properties of the solution. World Scientific Publication, Singapore.

[5] Cushing, J. M., Constantino, R. F., Brian, D.; Desharnais, R. A.; Shandeke, M. A. (1998).
Nonlinear Dynamics: Models, Experiment and Data, J. Theor boil. 194, 1 – 9.

[6] John, H. M. and Kurtis, D. K. (1994).
Using MATLAB as a programming language for numerical analysis, Int. J. Math. Educ. Sci Tech., Vol, 25, No. 4, 481 – 490.

[7] May, R. M. (1974)
Stability and Complexity in model Ecosystems, Princeton University Press, Princeton, NJ.

[8] Lasksmikantham, V.; Bainov, D. D. and Simeonov, P. S. (1989)
Theory of impulsive Differential Equations (Singapore: World Scientific Publications).

[9] Naylor Thomas H., Joseph L. Balinfty D., B and Kongehuk, K. (1966)
Computer Simulation Techniques John Wiley and Son Inc. U.S.A.

[10] Neclamkawel Francis (1987)
Computer Simulation and Modeling John Wiley and Sons Inc. U.S.A.

[11] Oyelami, B. O. (1999).
On military model for impulsive reinforcement functions using exclusion and marginalization techniques Nonlinear Analysis 35, 947 – 958.

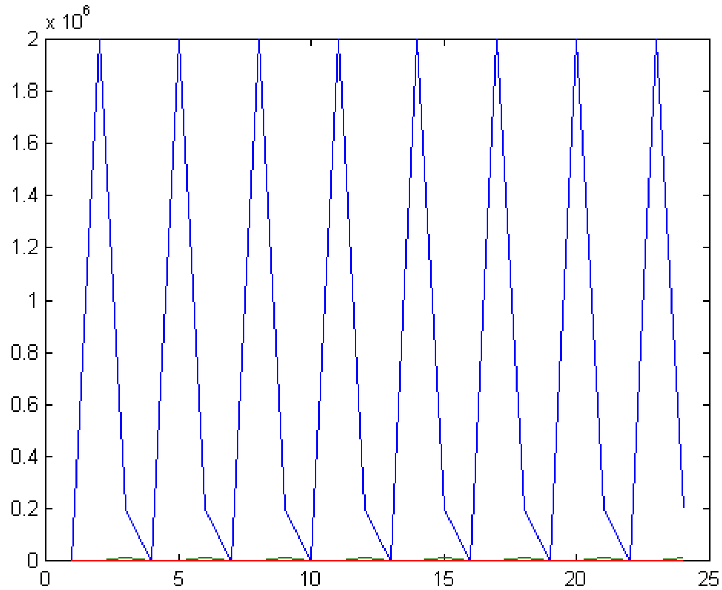
[12] Oyelami, B. O. and Ale, S. O. (2002)
B-transform and its application to a Fish-Hyacinth model. Int. J. Math. Edu. Sci. Tech.. Vol. 33, No. 4, 565 – 573.

[13] Pandit, S. G. and Deo Sudashiv, G. (1980).
Differential system involving impulses, Lecture Notes in Mathematic. Springer-Velag, Berlin – Heidelberg – New York.

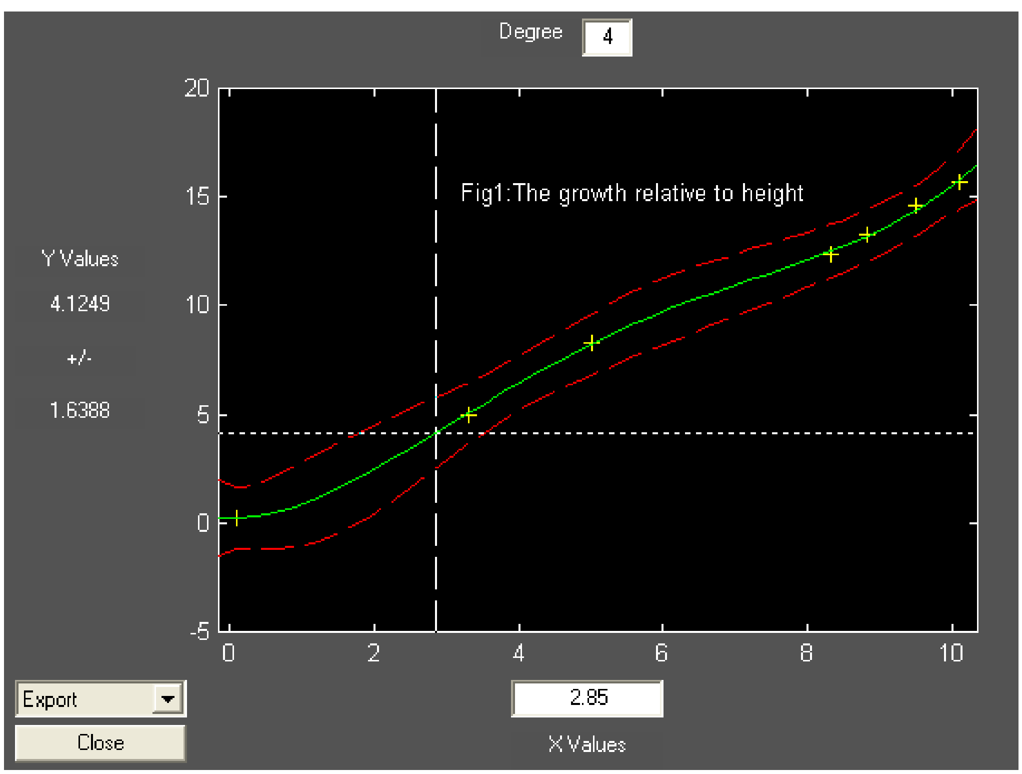
[14] Kar and Chaudhri (2002)
Bionomic of Multi-Species fish and optimal harvesting. Int. J. Math. Edu. Sci. Tech., Vol. 35, No. 4.

[15] Tilman D.; May, R. M.; Lehemen, C. L.; Nowak, M. A. (1994).
Habitat destruction and the extinction debt. Nature (London) 377, 65 – 66.

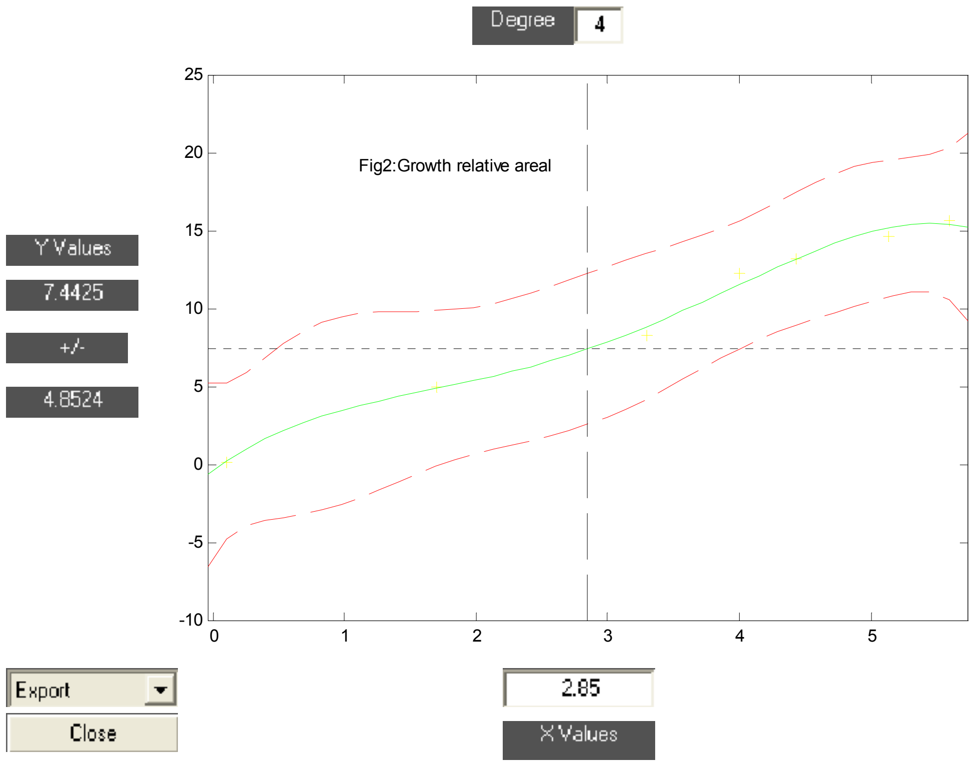
Appendix

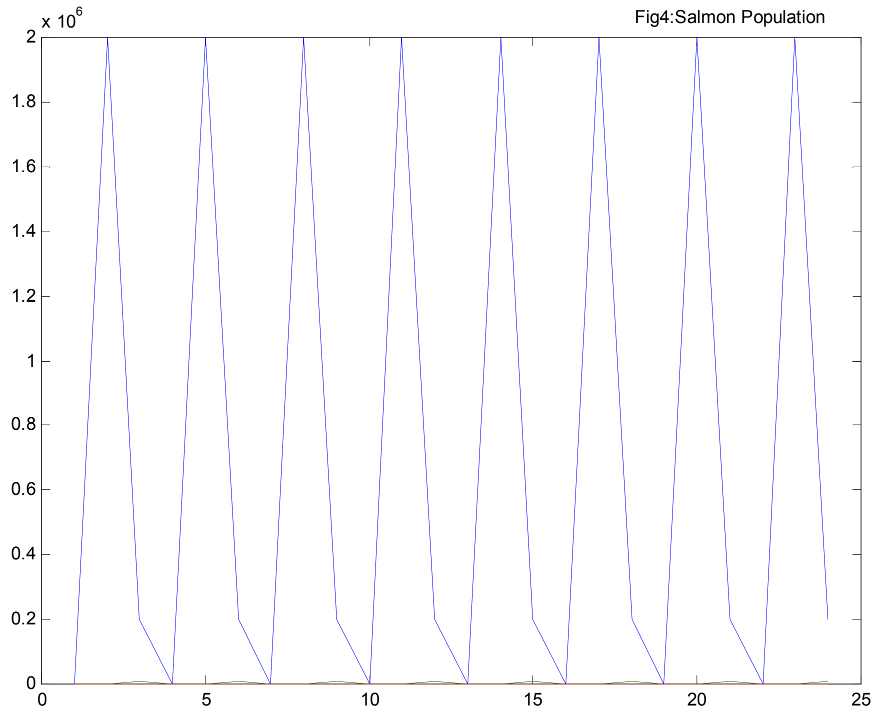


x 10^6
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25



Degree
4
20
15
10
5
0
-5
Fig1:The growth relative to height
Y Values
4.1249
+/-
1.6388
0
2
4
6
8
10
Export
Close
2.85
X Values


Degree
4
Y Values
7.4425
+/-
4.8524
Fig2:Growth relative areal
25
20
15
10
5
0
-5
-10
0
1
2
3
4
5
Export
Close
2.85
X Values


x 10^6
Fig4:Salmon Population
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25